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ON SYMPLECTIC PERIODS OF THE DISCRETE SPECTRUM OF GL_{2n}

BY

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ABSTRACT

We provide a formula for the symplectic period of an automorphic form in the discrete spectrum of GL_{2n}. It is a generalization of a formula of Jacquet and Rallis.

1. Introduction

Let G be a connected reductive group defined over a number field F, and let H be the fixed point subgroup of an involution on G. Denote by $\mathbb{A}$ the ring of adèles of F. Let φ be an automorphic form on $G(\mathbb{A})$. If φ is a cusp form then the period integral

$$l_H(\varphi) = \int_{H(F)\backslash(H(\mathbb{A})\cap G(\mathbb{A})^1)} \varphi(h)dh$$

is convergent by [AGR93]. For a more general automorphic form, the period integral may not converge and it is of interest to define $l_H(\varphi)$ via a regularization. See the introduction of [LR03] for a discussion and motivation. The case where E/F is a quadratic extension, H is a connected reductive group defined over F and $G = \mathrm{Res}_{E/F}H$, is referred to as the Galois case. A regularization of the period integral was introduced in [JLR99] in the split Galois case, i.e. when H is split over F. A general treatment of the Galois case was then given in [LR03]. The regularized period of an Eisenstein series is computed in terms of the so-called intertwining periods ([LR03], Theorem 9.1.1). This result is then

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used to obtain a formula for the (convergent) period integral of a truncated Eisenstein series. The formula, obtained in ([LR03], Proposition 11.1.1), is a relative analogue of the Maass–Selberg relations.

In this paper we consider a specific non-Galois case, namely, the case where $G = GL_{2n}$ and H is the symplectic group Sp_{2n}. We then call $l_H(\varphi)$ the symplectic period of φ. Our main result is a formula for the symplectic period of an automorphic form in the discrete spectrum of $G(\mathbb{A})$. It generalizes a formula of Jacquet and Rallis [JR92b]. We refer to the body of the work for any unexplained notation in the description below.

The discrete spectrum of $G(\mathbb{A})$ is described by Mœglin and Waldspurger [MW89]. An irreducible unitary representation of $G(\mathbb{A})$ is called a discrete automorphic representation of G if it occurs as a discrete summand in the space $L^2(G(F)\backslash G(\mathbb{A})^1)$. There is a bijection between discrete automorphic representations π of $G(\mathbb{A})$ and pairs (r,τ) where r divides $2n$ and τ is an irreducible cuspidal automorphic representation of $GL_r(\mathbb{A})$. Given such a pair (r,τ), let $2n = sr$ and let $P = MU$ be the standard parabolic subgroup of G of type $(r,\dots,r)$. The representation π is the unique irreducible quotient of the representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(|\det|^{\frac{s-1}{2}}\tau \otimes |\det|^{\frac{s-3}{2}}\tau \otimes \cdots \otimes |\det|^{\frac{1-s}{2}}\tau) \tag{1}$$

unitarily induced from $P(\mathbb{A})$ to $G(\mathbb{A})$. Let $E(\varphi,\lambda)$ be the cuspidal Eisenstein series induced from $P(\mathbb{A})$, as defined in §6 for a suitable section φ in the induced representation space. The Eisenstein series $E(\varphi,\lambda)$ is meromorphic in the complex parameter $\lambda = (\lambda_1,\dots,\lambda_s) \in \mathbb{C}^s$ and

$$\left[\prod_{i=1}^{s-1}(\lambda_i - \lambda_{i+1} - 1)\right] E(\varphi,\lambda) \tag{2}$$

is holomorphic at the point

$$\Lambda = \Big(\frac{s-1}{2}, \frac{s-3}{2}, \dots, \frac{1-s}{2}\Big).$$

We define the multi-residue of the Eisenstein series $E_{-1}(\varphi)$ to be the limit of (2) as $\lambda \to \Lambda$. The functions $E_{-1}(\varphi)$ are L^2-automorphic forms. As φ ranges over (1), the multi-residues $E_{-1}(\varphi)$ form an irreducible representation of $G(\mathbb{A})$ (see [Jac84]). This is the representation π corresponding to (r,τ). To compute symplectic periods of automorphic forms in the discrete spectrum, we are therefore reduced to the study of the symplectic period of $E_{-1}(\varphi)$. This period is given by the absolutely convergent integral

$$\int_{H(F)\backslash H(\mathbb{A})} E_{-1}(h,\varphi)dh$$

(see Lemma 40). A mixed truncation operator Λ_m^T defined on automorphic forms ϕ on $G(\mathbb{A})$ was introduced in [JLR99] for the Galois case. We define the mixed truncation similarly in our (non-Galois) case. It is a variant of Arthur's truncation operator Λ^T that is well adapted for the computation of periods. For a sufficiently regular parameter T, $\Lambda_m^T\phi$ is rapidly decreasing on $H(\mathbb{A})$. To compute the symplectic period of $E_{-1}(\varphi)$, we use a formula for the convergent period integral

$$\int_{H(F)\backslash H(\mathbb{A})} \Lambda_m^T E(h,\varphi,\lambda)dh.$$

The formula for the symplectic period of a truncated Eisenstein series, Theorem 7.5, is the relative analogue of the Maass–Selberg relations for our case. To obtain Theorem 7.5, we follow closely the guidelines of the proof of Proposition 11.1.1 in [LR03]. Many of the partial results there apply almost word by word in our case. Some of those results are quoted in this text without proof. For others we remark about the slight modifications required to adapt the proofs of Lapid and Rogawsky. To proceed with the computation of the period of $E_{-1}(\varphi)$, we observe that as in the Galois case, also here for an automorphic form ϕ on $G(\mathbb{A})$, the function of T defined for T sufficiently positive by the integral

$$\int_{H(F)\backslash H(\mathbb{A})} \Lambda_m^T\phi(h)dh$$

is an exponential polynomial function, i.e. it equals $\sum p_\lambda(T)e^{\langle\lambda,T\rangle}$ for some finite set of $\lambda\in\mathbb{C}^s$ and polynomials p_λ. Denote by $\mathcal{A}_0(G)$ the space of automorphic forms for which the polynomial p_0 is a constant. We show that $\phi=E_{-1}(\varphi)$ lies in $\mathcal{A}_0(G)$ and that its symplectic period is given by this constant. We then use the relative Maass–Selberg relations to compute the zero coefficient.

For every permutation w on $\{1,\dots,s\}$ denote by $M(w,\lambda)$ the standard intertwining operator on the space of automorphic forms on $U(\mathbb{A})M(F)\backslash G(\mathbb{A})$. Denote by $M_{-1}(w)$ the multi-residue at $\lambda=\Lambda$ of $M(w,\lambda)$. It is defined as in (2). For an automorphic form φ on $U(F)M(\mathbb{A})\backslash G(\mathbb{A})$ define

$$j(\varphi)=\int_{K\cap H(\mathbb{A})}\int_{(M\cap H)(\mathbb{A})^1}\varphi(mk)dmdk.$$

Our main result is

THEOREM 1.1: *Let φ be an automorphic form on $U(\mathbb{A})M(F)\backslash G(\mathbb{A})$ that lies in the space (1). If s is odd, then*

$$\int_{H(F)\backslash H(\mathbb{A})} E_{-1}(h,\varphi)dh = 0.$$

If $s = 2k$ is even, then

$$\int_{H(F)\backslash H(\mathbb{A})} E_{-1}(h,\varphi)dh = v_{P_H} j(M_{-1}(w_\sigma)\varphi)$$

where v_{P_H} is a certain volume, σ is any permutation on $\{1,\dots,k\}$ and w_σ is the permutation given by

$$w_\sigma(2i-1) = \sigma^{-1}(i), \quad w_\sigma(2i) = s+1-\sigma^{-1}(i), \quad i = 1,\dots,k.$$

The apparently non-canonical formula for the period (the freedom in choosing σ) is interpreted in a canonical form in §8. It is the multi-residue $J_{-1}(\xi_0,\varphi)$ at $\lambda = \Lambda$ of an intertwining period at a twisted involution ξ_0 which is represented by each of the permutations w_σ.

When $s \le 2$, the result was proved in [JR92b]. The vanishing of the symplectic period of a cusp form ($s = 1$) follows from local results of [HR90].

In fact, whenever s is even, there is an automorphic form in the space of π with a non-vanishing symplectic period. In the case $s = 2$, this is the content of Proposition 2 in [JR92b]. In that case the permutation w_1, defined in Theorem 1.1, is the identity and the period is simply given by $j(\varphi)$. For this reason, the proof of the non-vanishing is easy. For a general even s, a proof of the non-vanishing of the period is more complicated. In [Off], we provide the proof and therefore determine precisely which discrete automorphic representations of $G(\mathbb{A})$ have a symplectic period.

The rest of the work is organized as follows. In §2 we set up notation. In §3 we provide a careful study of the double coset space $P\backslash G/H$ for a parabolic subgroup P of G, based on the theory of twisted involutions established in [Spr85] and in [LR03]. This study is essential both for the proof of Theorem 7.5 and of Theorem 1.1. Another important concept we need for applying the proof of [LR03] to Theorem 7.5 is that of intertwining periods. We introduce them in §4, where we also state the main results regarding their convergence, and follow the guidelines of [LR03] to reduce the proof of convergence to a special case. In §5 we generalize an integration formula of Jacquet and Rallis and use it to prove the convergence in this special case. In §6 we obtain a distributional formula for the period of a pseudo-Eisenstein series, which we use in §7 to complete the proof of Theorem 7.5. Section 8 is the heart of the paper. It applies the relative Maass–Selberg relations to the proof of Theorem 1.1.

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2. Notation

Let F be a number field and let $\mathbb{A}$ be the ring of adèles of F. For an algebraic group X defined over F we will often write X also for the group $X(F)$ of rational points. We will denote by δ_X the modulus function on $X(\mathbb{A})$. Throughout most of this work G will denote the group GL_{2n}. For some inductive arguments in §7, G will denote a standard Levi thereof. Thus we set up the following notation for any group G of the form $G = GL_{n_1} \times \cdots \times GL_{n_s}$ with $2n = n_1 + \cdots + n_s$. Let $P_0 = T_0U_0$ be the Borel subgroup of G consisting of the upper triangular matrices in G, where T_0 is the group of diagonal matrices and U_0 the unipotent radical of P_0. There is also a standard choice of a maximal compact of $G(\mathbb{A})$ which we denote by K.

By a parabolic subgroup of G we will always mean a standard parabolic, i.e. one that contains P_0. Similarly, a Levi subgroup will mean a Levi subgroup of a standard parabolic, which contains T_0. We will always reserve the letters P, Q for parabolic subgroups with Levi decompositions

$$P = MU, \quad Q = LV,$$

with Levi subgroups M, L and unipotent radicals U, V. For a parabolic subgroup $P = MU$ of G, set

$$\mathfrak{a}_M^* = X^*(M) \otimes_{\mathbb{Z}} \mathbb{R}$$

where $X^*(\cdot)$ is the lattice of rational characters of an algebraic group. Denote the dual space by $\mathfrak{a}_M$. We will also denote $\mathfrak{a}_M$ by $\mathfrak{a}_P$ and $\mathfrak{a}_{P_0}$ by $\mathfrak{a}_0$. We use similar notation for the dual spaces. For Levi subgroups $M \subset L$ there is a canonical direct sum decomposition

$$\mathfrak{a}_M = \mathfrak{a}_L \oplus \mathfrak{a}_M^L.$$

A similar decomposition holds for the dual spaces. We define a height function

$$H_M : G(\mathbb{A}) \to \mathfrak{a}_M.$$

It is the left $U(\mathbb{A})$-invariant, right K-invariant function on $G(\mathbb{A})$ such that for $m \in M(\mathbb{A})$,

$$e^{\langle \chi, H_M(m)\rangle} = |\chi|(m) = \prod_{\nu} |\chi_\nu(m_\nu)|_\nu$$

for all $\chi \in X^*(M)$. Here, χ_ν is the extension of scalars of χ to the completion F_ν of F at ν, and the product is over all places ν of F. Denote

$$M(\mathbb{A})^1 = \bigcap_{\chi \in X^*(M)} \mathrm{Ker}|\chi|.$$

The function H_M defines an isomorphism $M(\mathbb{A})^1 \backslash M(\mathbb{A}) \simeq \mathfrak{a}_M$. We write H_0 for H_{T_0}. The embedding

$$\mathbb{R} \hookrightarrow F \otimes_{\mathbb{Q}} \mathbb{R} = F_\infty \hookrightarrow \mathbb{A}$$

given by $x \mapsto 1 \otimes x$ defines a subgroup A_0 of $T_0(\mathbb{A})$ which is isomorphic to $(\mathbb{R}_+^*)^{2n}$. For every Levi subgroup M of G we denote by T_M the intersection of T_0 with the center of M and by A_M the intersection of A_0 with the center of M. Then $M(\mathbb{A}) = A_M M(\mathbb{A})^1$. There is an isomorphism $A_M \simeq \mathfrak{a}_M$ which we denote by $e^X \leftrightarrow X, X \in \mathfrak{a}_M$.

2.1 Roots and co-roots. For a Levi subgroup M let $R(T_0, M)$ denote the set of roots of T_0 in M. It is a subset of $(\mathfrak{a}_0^M)^*$. The parabolic subgroup $P_0 \cap M$ of M determines sets Δ_0^M and $R^+(T_0, M)$ of simple roots and positive roots respectively. For Levi subgroups $M \subset L$ let Δ_M^L denote the set of non-zero restrictions of elements of Δ_0^L to $\mathfrak{a}_M^L$. Thus Δ_M^L spans $(\mathfrak{a}_M^L)^*$. We make similar definitions for co-roots in the dual spaces. Thus, $(\Delta^\vee)_M^L$ spans $\mathfrak{a}_M^L$. The pairing on $\mathfrak{a}_0^* \times \mathfrak{a}_0$ is denoted by $\langle \cdot, \cdot \rangle$. It induces a non-degenerate pairing on $(\mathfrak{a}_M^L)^* \times \mathfrak{a}_M^L$. Let $(\hat{\Delta})_M^L$ be the dual basis of $(\Delta^\vee)_M^L$ in $(\mathfrak{a}_M^L)^*$, and let $(\hat{\Delta}^\vee)_M^L$ be the dual basis of Δ_M^L in $\mathfrak{a}_M^L$. Let $\rho_0 \in \mathfrak{a}_0^*$ be half the sum of the positive roots $R^+(T_0, G)$. Let ρ_P be the projection of ρ_0 on $\mathfrak{a}_M^*$. The modulus function of $P(\mathbb{A})$ is then given by

$$\delta_P(\cdot) = e^{\langle 2\rho_P, H_M(\cdot)\rangle}.$$

2.2 Weyl groups. Throughout this work, we will identify the permutation group $\mathfrak{S}_r$ of $\{1,\dots,r\}$ with the $r\times r$ permutation matrices, thus $\tau\in\mathfrak{S}_r$ is both a bijection of $\{1,\dots,r\}$ with itself and the $r\times r$ matrix $(\delta_{i,\tau(j)})$.

For M a Levi subgroup of G of type $(m_1,\dots,m_s)$, the Weyl group W_M of M is identified with $\mathfrak{S}_{m_1}\times\cdots\times\mathfrak{S}_{m_s}$. We denote $W=W_G$. For Levi subgroups $M,M_1\subset L$ we denote by $W_L(M,M_1)$ the set of elements $w\in W_L$ of minimal length in wW_M such that $wMw^{-1}=M_1$. Set

$$W_L(M)=\bigcup_{M_1}W_L(M,M_1).$$

We set $W(M,M_1)=W_G(M,M_1)$ and $W(M)=W_G(M)$. The length function $l_M\colon W(M)\to\mathbb{Z}_{\geq 0}$ is defined in [MW94,§I.1.7] by

$$l_M(w)=\#\{\alpha\in R^+_{ind}(T_M,G)|w\alpha<0\}.$$

For $\alpha\in R^+(T_M,G)$, we will denote by s_α the unique $w\in W(M)$ such that $l_M(w)=1$ and $w\alpha<0$. Set $l=l_{T_0}$. If $M\subset L$ we write w_M^L for the longest element in $W_L(M)$. We will denote $w_0^L=w_{T_0}^L$ and let $w_0=w_0^G$ be the longest element of W. Finally, set

$$(\mathfrak{a}_M^*)_+=\{X\in\mathfrak{a}_M^*|\langle X,\alpha^\vee\rangle>0\text{ for all }\alpha\in\Delta_M\}$$

to be the positive Weyl chamber of $\mathfrak{a}_M^*$.

2.3 Bruhat decomposition. There is a bijection $P_0\backslash G/P_0\simeq W$ given by $P_0wP_0\leftrightarrow w$. More generally, there is a bijection $Q\backslash G/P\simeq W_L\backslash W/W_M$ for any two parabolic subgroups Q,P. Let ${}_LW_M$ be the set of reduced representatives, i.e. of elements of minimal length in the double cosets of $W_L\backslash W/W_M$. There is a bijection

$$W_L\backslash W/W_M\simeq {}_LW_M$$

which we use to identify $W_L\backslash W/W_M$ with the set of reduced elements. We further denote

$${}_LW_M^c=\{w\in {}_LW_M|wMw^{-1}\subset L\}.$$

If $w\in {}_LW_M^c$, then wMw^{-1} is a (standard) Levi subgroup of L.

2.4 Measures. Identifying $\mathfrak{a}_0$ with $\mathbb{R}^{2n}$ we may use the standard scalar product to determine a norm $\|\|$ on $\mathfrak{a}_0$, which gives a Haar measure on $\mathfrak{a}_0$. On the dual space $\mathfrak{a}_0^*$ we choose a Haar measure which is dual with respect to the Fourier transform. The inner product also determines a Haar measure on the subspaces. We get a Haar measure on A_M through its isomorphism with $\mathfrak{a}_M$. Discrete groups are equipped with the counting measures. For a unipotent group U we use the Haar measure that gives $\mathrm{Vol}(U\backslash U(\mathbb{A})) = 1$. We also fix a Haar measure dk that gives K total volume 1. We fix a Haar measure dg on $G(\mathbb{A})$. For a Levi subgroup M of G, a Haar measure dm on $M(\mathbb{A})$ is then determined by

$$\int_{G(\mathbb{A})} f(g)dg = \int_{U(\mathbb{A})\times M(\mathbb{A})\times K} f(umk)e^{-\langle 2\rho_P, H_M(m)\rangle} dudmdk.$$

2.5 THE SYMMETRIC SPACE. Let w_n be the $n \times n$ permutation matrix with unit anti-diagonal, and let

$$\epsilon = \epsilon_{2n} = \begin{pmatrix} & w_n \\ -w_n & \end{pmatrix}.$$

We define the involution θ on G by

$$\theta(g) = \epsilon\, {}^t g^{-1} \epsilon^{-1}.$$

The symmetric space attached to (G, θ) is the variety

$$\mathcal{C} = \mathcal{C}_G(\theta) = \{x \in G | x\theta(x) = 1_{2n}\}.$$

The group G acts on $\mathcal{C}$ by the twisted conjugation

$$g \star x = g \star_\theta x = gx\theta(g)^{-1}.$$

Until it is otherwise specified, set $G = GL_{2n}$. We observe that $\mathcal{C}\epsilon$ is the set of skew-symmetric matrices in G. Therefore $\mathcal{C}$ is a unique G-orbit. For a subgroup Q of G we will denote by Q_x the stabilizer of x in Q. However, we will denote by H_x the group G_x and further by $H = H_{2n}$ the stabilizer in G of the identity. For each $x \in \mathcal{C}$, the group H_x is the symplectic group obtained from the skew-symmetric form defined by $(x\epsilon)^{-1}$. We will denote by θ_x the involution sending $g \in G$ to $x\theta(g)x^{-1}$. Thus, the set G^{θ_x} of θ_x-fixed points of G coincides with H_x and

$$\mathcal{C}_G(\theta_x) = \mathcal{C}\, x^{-1}. \tag{3}$$

If $\eta \in G$ is such that $x = \eta\theta(\eta)^{-1}$, then $H_x = \eta H \eta^{-1}$. We remark that (G, θ) is a relatively quasi-split pair, in the sense of [LR03], i.e. θ stabilizes P_0. For a subgroup Q of G we will always denote $Q_H = Q \cap H$. The group $(P_0)_H$ is a Borel subgroup of H with Levi decomposition $(P_0)_H = (T_0)_H(U_0)_H$. With respect to $(P_0)_H$ we can speak of standard parabolic subgroups of H. We will keep our convention and refer to a standard parabolic subgroup of H simply as a parabolic and to a standard Levi subgroup of H simply as a Levi. Note that θ maps a parabolic of G of type $(n_1,\dots,n_t)$ to a parabolic of type $(n_t,\dots,n_1)$. There is a one to one correspondence between θ-stable parabolics of G and parabolics of H. If $Q=LV$ is a θ-stable parabolic, then $Q_H=L_HV_H$ is a parabolic of H with Levi subgroup L_H and unipotent radical V_H.

2.6 Root, co-root spaces and measures for H. The map θ stabilizes P_0 and therefore defines an involution on $\mathfrak{a}_0$. For $x\in\mathfrak{a}_0$ we denote by x_θ^+ (resp. x_θ^-) the projection of x onto the 1-eigenspace (resp. -1-eigenspace). We use similar notation for the dual space. We identify the space $(\mathfrak{a}_0)_\theta^+$ with $X^*((T_0)_H)\otimes_{\mathbb{Z}}\mathbb{R}$. For θ-stable parabolic subgroups $P\subset Q$ of G we define $\Delta_{P_H}^{Q_H}=(\Delta_P^Q)_\theta^+\backslash\{0\}$; then $\Delta_{P_H}^{Q_H}$ spans $(\mathfrak{a}_P^Q)_\theta^+$. The set $\Delta_{(P_0)_H}=\Delta_{(P_0)_H}^H$ forms a basis of simple roots for H with respect to the Borel subgroup $(P_0)_H$ of H. We make similar definitions in the spaces of co-roots and denote by $(\hat{\Delta}^\vee)_{P_H}^{Q_H}$ the dual basis of $\Delta_{P_H}^{Q_H}$ in $((\mathfrak{a}^*)_P^Q)_\theta^+$ and by $(\hat{\Delta})_{P_H}^{Q_H}$ the dual space of $(\Delta^\vee)_{P_H}^{Q_H}$ in $(\mathfrak{a}_P^Q)_\theta^+$. Our convention about Haar measures on $H(\mathbb{A})$ and its subgroups are analogous to those for $G(\mathbb{A})$. The measure on $(\mathfrak{a}_0)_\theta^+$ is given by that on $\mathfrak{a}_0$.

3. Double cosets

Our goal in this section is to analyze, for any parabolic subgroup P of G, the set $P\backslash\mathcal{C}$ of P orbits in $\mathcal{C}$. We will use the notion of twisted involutions developed in [Spr85] and further extended in [LR03] in connection with the relative trace formula. We therefore start by repeating definitions and some pertinent results from §3 of [LR03].

3.1 Twisted involutions. Twisted involutions are defined with respect to an involution σ of $\mathfrak{a}_0$ that maps Δ_0 to itself. Since θ stabilizes P_0 it acts on $\mathfrak{a}_0$. Identifying $\mathfrak{a}_0$ with $\mathbb{R}^{2n}$ the action can be described explicitly as

$$\theta(x_1,\dots,x_{2n})=(-x_{2n},\dots,-x_1). \tag{4}$$

Thus, θ preserves the set of simple roots. Therefore, the results of [LR03] hold in our case. In this section we shall set up the notation and quote the results of Lapid and Rogawski needed later, concerning twisted involutions.

Definition: A **twisted involution** is an element $\xi\in W$ such that $\theta(\xi)=\xi^{-1}$. Let $\mathfrak{I}_0(\theta)$ be the set of twisted involutions.

The linear map (4) of θ on $\mathfrak{a}_0$ induces on W the action $\theta(w) = w_0 w w_0$ given by conjugation by the longest element. The Weyl group W acts on $\mathfrak{I}_0(\theta)$ by

$$w * \xi = w\xi\theta(w)^{-1}.$$

We deliberately differentiate this action from the G-action $\star$ on $\mathcal{C}$ since, viewed as matrices in G, the matrix $w \star \xi$ may no longer be a permutation matrix (but a signed permutation matrix).

More generally, let $P = MU$ be a parabolic subgroup. A double coset D in $W_M \backslash W / W_{\theta M}$ satisfies $\theta(D) = D^{-1}$ if and only if the reduced representative of D is a twisted involution ([LR03], Lemma 3.1.1).

Definition: Let D be a double coset in $W_M \backslash W / W_{\theta M}$ with reduced representative ξ, such that $\theta(\xi) = \xi^{-1}$. We say that ξ is an **admissible twisted involution** if $\xi\theta(M)\xi^{-1} = M$. Let $\mathfrak{I}_M(\theta)$ denote the set of admissible twisted involutions.

If $\xi \in \mathfrak{I}_M(\theta)$ then $\xi\theta$ acts as an involution on $\mathfrak{a}_M^*$ and on $\mathfrak{a}_M$. Let $(\mathfrak{a}_M^*)_{\xi\theta}^{\pm}$ be the ± 1 eigenspaces of $\xi\theta$ in $\mathfrak{a}_M^*$. We use similar notation for the dual space.

Definition: An admissible twisted involution $\xi \in \mathfrak{I}_M(\theta)$ is called **minimal** if there exists a θ-stable Levi subgroup $L \supset M$ such that $\xi = w_{\theta M}^L$ and $\xi\theta\alpha = -\alpha$ for all $\alpha \in \Delta_M^L$. In this case L is uniquely determined by ξ and is denoted $L_{\xi,\theta}$. Let $\Xi_M(\theta)$ denote the set of minimal twisted involutions in $\mathfrak{I}_M(\theta)$.

From the definitions it follows that if $\xi \in \Xi_M(\theta)$ and $L = L_{\xi,\theta}$, then

$$(\mathfrak{a}_M^*)_{\xi\theta}^- = (\mathfrak{a}_M^L)^* \oplus (\mathfrak{a}_L^*)_\theta^- \tag{5}$$

and

$$(\mathfrak{a}_M^*)_{\xi\theta}^+ = (\mathfrak{a}_L^*)_\theta^+. \tag{6}$$

In ([LR03], §3.3) a directed graph was attached to an associated class of Levi subgroups, to describe the combinatorics of twisted involutions. For $\xi \in \mathfrak{I}_M(\theta)$ and $\xi' \in \mathfrak{I}_{M'}(\theta)$, the set $W(\xi, \xi')$ of paths on the graph and the set $W^0(\xi, \xi')$ of loop-free paths were defined. Lapid and Rogawski provided a useful characterization of those sets which we will use here as their definitions. This way, we avoid introducing notation we will not need. We set

$$W(\xi, \xi') =$$
$$\{w \in W(M, M') | w * \xi = \xi', w\beta > 0 \text{ for all } \beta \in R^+(T_M, G) \text{ such that } \xi\theta\beta = \beta\}$$

and

$$W^0(\xi,\xi') =$$
$$\{w \in W(M,M')|w*\xi = \xi', w\beta>0 \text{ for all } \beta \in R^+(T_M,G) \text{ such that } \xi\theta\beta = \pm\beta\}.$$

The following is the content of Corollary 3.4.1 in [LR03].

PROPOSITION 3.1: *For every $\xi \in \mathfrak{I}_M(\theta)$ there exists $\xi' \in \Xi_{M'}(\theta)$ and $w \in W^0(\xi,\xi')$.*

3.2 P_0-ORBITS. Let W^θ be the set of fixed points of θ in W and let

$$W(\theta) = W * 1.$$

Then, W^θ is the centralizer of w_0 and the map $w \mapsto w\theta(w)^{-1}$ defines an isomorphism $W/W^\theta \simeq W(\theta)$. Using the Bruhat decomposition, we define a map ι_0: $P_0\backslash\mathcal{C} \to W$. For $x \in \mathcal{C}$ and $\mathcal{O} = P_0 \star x$, let $\iota_0(\mathcal{O}) = \xi \in W$ where $P_0xP_0 = P_0\xi P_0$. We will view ι_0 as a map either from $\mathcal{C}$ or from its P_0-orbits. The following proposition differs from its analogue in the Galois case. While in the Galois case, the image of ι_0 is the entire set of twisted involutions, in the case at hand, the image is a unique Weyl orbit.

PROPOSITION 3.2: *The map ι_0 is a bijection $P_0\backslash\mathcal{C} \simeq W(\theta)$.*

Proof: Let $x \in \mathcal{C}$ and denote $\mathcal{O} = P_0 \star x$. For $a \in T_0, w \in W$ we denote ${}^wa = waw^{-1}$. In [JR92a] it is shown that if ${}^tX = -X$ is a non-singular skew-symmetric matrix, then there is $u \in U_0$ such that $X = uaw{}^tu$, where $w^2 = 1$ and ${}^wa = -a$. Let $x \in \mathcal{C}$; then for $x\epsilon$ there exist a, w, u as above, thus

$$x = uaw{}^tu\epsilon^{-1} = uaw\epsilon^{-1}\theta(u)^{-1}.$$

We therefore see that $aw\epsilon^{-1} \in T_0ww_0 \cap \mathcal{O}$ and hence $\iota_0(x) = ww_0$. If $a = \mathrm{diag}(a_1,\dots,a_{2n})$ then ${}^wa = \mathrm{diag}(a_{w^{-1}(1)},\dots,a_{w^{-1}(2n)})$. Any permutation of order two can be expressed as a product of disjoint reflections $w = (i_1\ j_1)\cdots(i_r\ j_r)$ with $r \le n$. Since ${}^wa = -a$ we get that $a_i = -a_{w(i)}$ for all i, which shows that w has no fixed points, i.e. that $r = n$ and thus w is conjugate to w_0, which is the same as saying that $ww_0 \in W(\theta)$. This proves that ι_0 is into $W(\theta)$ and that any Borel orbit in $\mathcal{C}$ intersects T_0W. If $x, y \in \mathcal{C}$ are such that $\iota_0(x) = \iota_0(y) = ww_0$, then up to twisted conjugation by an element of U_0 we may assume $x = aw\epsilon^{-1}, y = bw\epsilon^{-1}$ with $a, b \in T_0$ such that ${}^wa = -a, {}^wb = -b$ and $w = (i_1\ j_1)\cdots(i_n\ j_n)$ lies in the conjugacy class of w_0. If $c = \mathrm{diag}(c_1,\dots,c_{2n})$

with $c_{i_k} = b_{i_k}$ and $c_{j_k} = a_{i_k}^{-1}$, then $ca^w c = b$. Therefore, $c \star (aw\epsilon^{-1}) = bw\epsilon^{-1}$. This shows that ι_0 is injective. To show the map is surjective, for an element $\xi \in W(\theta)$, we observed already that ξw_0 is conjugate to w_0 and therefore is a product of n disjoint reflections. Denote $\xi w_0 = (i_1\ j_1) \cdots (i_n\ j_n)$. We denote by $a_0 \in T_0$ the diagonal matrix such that $\epsilon = a_0 w_0$. Let $b = \mathrm{diag}(b_1, \ldots, b_{2n})$ with $b_{i_k} = 1, b_{j_k} = -1, k = 1, \ldots, n$, and $a = b^\xi a_0^{-1}$; then $a\xi \in \mathcal{C}$. ∎

COROLLARY 3.3: *The map $\mathcal{O} \mapsto \mathcal{O} \cap T_0 W$ defines a bijection*

$$P_0 \backslash \mathcal{C} \simeq T_0 \backslash (\mathcal{C} \cap T_0 W).$$

3.3 P-ORBITS. Let $P = MU$ be a parabolic subgroup of G. Using the Bruhat decomposition, we define a map $\iota_M\colon P\backslash\mathcal{C} \to {}_M W_{\theta(M)}$ sending a P-orbit $\mathcal{O} = P \star x$ in $\mathcal{C}$ to $\xi \in {}_M W_{\theta(M)}$, where $P\xi\theta(P) = Px\theta(P)$.

We observe that $W_{\theta(M)} = w_0 W_M w_0$, and therefore the map $D \mapsto Dw_0$ defines a bijection $W_M\backslash W / W_{\theta(M)} \simeq W_M \backslash W / W_M$ that takes the double coset of w to the double coset of ww_0. Thus it maps the twisted involutions to involutions, i.e. to Weyl elements of order two. In particular, a double coset containing an element of $W(\theta)$ is mapped to a double coset containing a conjugate of w_0. Note that since $l(ww_0) = l(w_0) - l(w)$, the reduced element will map to an element of maximal length in the double coset in the image. Hence double cosets in $W_M\backslash W/W_M$ that are involutions have elements of maximal length. We will refer to this map as **the dictionary** between twisted-involutions and involutions.

LEMMA 3.4: *Let L be a Levi subgroup of G and $\xi \in \mathfrak{I}_L(\theta)$. Assume that $W_L\xi \cap W(\theta)$ is non-empty. Then*
(1) $\xi \in W(\theta)$,
(2) $W_L\xi \cap W(\theta) = W_L * \xi$.

Proof: We use our dictionary to translate part (1) of the lemma in terms of involutions. It is equivalent to the statement: Let $\sigma \in W$ be an involution that normalizes L and is longest in $W_L\sigma$. If $W_L\sigma$ contains a conjugate of w_0 then σ is conjugate to w_0. Let $\mathfrak{n} = (n_1, \ldots, n_t)$ be the type of the Levi L. We set some notation to denote certain permutations that conjugate L to a Levi subgroup of G. If $\tau \in \mathfrak{S}_t$ and $\sigma_k \in \mathfrak{S}_{n_k}, k = 1, \ldots, t$ then we define the permutation

$$w_{\mathfrak{n}}(\tau; \sigma_1, \ldots, \sigma_t) \tag{7}$$

in W. In block form, it is the matrix $(A_{i,j})_{1 \le i,j \le t}$ where $A_{i,j}$ is the $n_i \times n_{\tau(j)}$ zero matrix unless $i = \tau(j)$, in which case $A_{i,j} = \sigma_i$. There is an involution

$\tau \in \mathfrak{S}_t$ such that

$$W_L\sigma = \{w_{\mathfrak{n}}(\tau;\sigma_1,\ldots,\sigma_t)|\sigma_k \in \mathfrak{S}_{n_k}, k=1,\ldots,t\}$$

and

$$\sigma = w_{\mathfrak{n}}(\tau; w_{n_1},\ldots,w_{n_t}).$$

An involution of the form (7) in $W_L\sigma$ will satisfy

$$\sigma_{\tau(k)} = \sigma_k^{-1}. \tag{8}$$

Let $\sigma' = w(\tau;\sigma_1,\ldots,\sigma_t)$ be a conjugate of w_0 in $W_L\sigma$. A permutation of order two in W is conjugate to w_0 if and only if it has no fixed points. Thus we must show σ has no fixed points. In other words, we must show the diagonal entries of σ are non-zero. In block form, the $(k,\tau(k))$-blocks $k=1,\ldots,t$ of both σ and σ' are the only non-zero ones. Thus a non-zero diagonal entry of σ can only appear when $k_0 = \tau(k_0)$. By (8), for such k_0 the block σ_{k_0} is an involution. However, since the diagonal entries of σ' are zero, so are the diagonal entries of σ_{k_0}. Therefore σ_{k_0} is an involution with no fixed points, which implies that n_{k_0} is even. Thus, the diagonal entries of σ in the (k_0,k_0)-block are zero and part (1) of the lemma follows. Since $W(\theta)$ is the W-orbit of the identity, it is now clear from (1) and from the fact that ξ is admissible that $W_L\xi \cap W(\theta) \supset W_L * \xi$. To show the other inclusion we again use our dictionary and prove the equivalent problem for involutions. Thus, if $\sigma = w_{\mathfrak{n}}(\tau; w_{n_1},\ldots,w_{n_t})$ and $\sigma' = w_{\mathfrak{n}}(\tau;\sigma_1,\ldots,\sigma_t)$ satisfies (8) and σ' (and hence also σ) is conjugate to w_0, we must show that we may conjugate σ' to σ with an element of W_L. An element of W_L may be written as $w = w_{\mathfrak{n}}(1;\nu_1,\ldots,\nu_t)$ with $\nu_k \in \mathfrak{S}_{n_k}$. Thus,

$$w\sigma' w^{-1} = w_{\mathfrak{n}}(\tau; \nu_i\sigma_i\nu_{\tau(i)}^{-1}).$$

We may write the involution τ as a product of disjoint permutations, say $\tau = (i_1j_1)\cdots(i_rj_r)$. If $\tau(i) = i$ we have already observed that σ_i must be conjugate to w_{n_i}; we then fix ν_i such that $\nu_i\sigma_i\nu_i^{-1} = w_{n_i}$. We also set $\nu_{i_k} = \sigma_{i_k}^{-1}$ and $\nu_{j_k} = w_{n_k}$ for all $k=1,\ldots,r$. In light of (8) we observe that we then have $w\sigma' w^{-1} = \sigma$, as desired. ∎

Proposition 3.5: *The map ι_M defines a bijection $P\backslash\mathcal{C} \simeq W(\theta) \cap {}_MW_{\theta(M)}$.*

Proof: We first show that the image of ι_M indeed lies in $W(\theta)$. Let $\mathcal{O} \in P\backslash\mathcal{C}$ and $\xi = \iota_M(\mathcal{O})$. We denote by D the double coset $W_M\xi W_{\theta(M)}$. If $x \in \mathcal{O}$ then $\iota_0(x) \in D$ and, by Proposition 3.2, $\iota_0(x) \in W(\theta)$. Denote $M' = M \cap \xi\theta(M)\xi^{-1}$.

It is a Levi subgroup of M. Let ξ' be an element of minimal length in $D \cap W(\theta)$. As an element of D it can be written uniquely as $\xi' = w_1 w' \xi w_2$, where $w' \in W_{M'}, w_1 \in W_M$ is right $W_{M'}$-reduced and $w_2 \in W_{\theta(M)}$ is left $W_{\theta(M')}$-reduced. This is a reduced expression for ξ' in the sense that $l(\xi') = l(w_1) + l(w') + l(\xi) + l(w_2)$. Since ξ' is a twisted involution we get that $w_2 = \theta(w_1)^{-1}$ and therefore $w'\xi$ is in $D \cap W(\theta)$. From the minimality of ξ' it follows that $\xi' = w'\xi$. This shows that $W_{M'}\xi \cap W(\theta)$ is not empty. Note that $\xi \in \mathfrak{I}_{M'}(\theta)$. Applying (1) of Lemma 3.4 with $L = M'$ we get that $\xi \in W(\theta)$. That ι_M is onto $W(\theta) \cap {}_M W_{\theta(M)}$ follows from Proposition 3.2. Indeed, if $\xi \in W(\theta) \cap {}_M W_{\theta(M)}$ then there is $\mathcal{O} \in P_0 \backslash \mathcal{C}$ such that $\iota_0(\mathcal{O}) = \xi$, therefore for each $x \in \mathcal{O}, \iota_M(P \star x) = \xi$. Now, let $\iota_M(P \star x) = \xi$. We have seen in the proof of Proposition 3.2 that $T_0 \xi \cap \mathcal{C}$ is non-empty. Let $y \in T_0 \xi \cap \mathcal{C}$. To prove injectivity it is enough to show that $P \star x = P \star y$. Let $\xi' = \iota_0(x)$. Replacing x by an element of $P_0 \star x$ we may assume that $x \in T_0 \xi' \cap \mathcal{C}$. In the first part of the proof it was shown that $\xi' = w * (w'\xi)$ for some $w \in W_M$ and $w' \in W_{M'}$. Replacing x by $w^{-1} \star x$ we may therefore assume that $\xi' = w'\xi$. From part (2) of Lemma 3.4 we have that the twisted involutions in $W_{M'}\xi$ form a unique $W_{M'}$-orbit. As before, let $w'\xi = w_1 * \xi$ with $w_1 \in W_{M'}$; then $w_1^{-1} \star x \in T_0 \xi \cap P \star x$. We see that $T_0 \xi$ intersects both $P \star x$ and $P \star y$. By Corollary 3.3, $T_0 \xi \cap \mathcal{C}$ is a unique T_0-orbit and therefore we get that $P \star x = P \star y$. ∎

Let $\xi \in W(\theta) \cap {}_M W_{\theta(M)}$. We set as before $M' = M \cap \xi\theta(M)\xi^{-1}$ and let $U' = M \cap \xi\theta(U)\xi^{-1}$. Then $P' = M'U'$ is a parabolic subgroup of M. In view of the previous proposition we may denote by $\mathcal{O}_\xi$ the unique P-orbit that ι_M maps to ξ. Fix $x_0 \in T_0 \xi \cap \mathcal{C}$. Then θ_{x_0} preserves the standard Borel subgroup of M' and induces on $\mathfrak{a}_{M'}$ the linear transform $\xi\theta$. We define the map

$$x \mapsto x' = x x_0^{-1}$$

from $\mathcal{C} \cap M'\xi$ to $\mathcal{C}_{M'}(\theta_{x_0})$.

PROPOSITION 3.6: *With the above notation,*

(1) *the map $x \mapsto x'$ defines a bijection*

$$\mathcal{C} \cap M'\xi \simeq \mathcal{C}_{M'}(\theta_{x_0})$$

which intertwines the M'-action of $\star_\theta$ with $\star_{\theta_{x_0}}$,

(2) *$\mathcal{O}_\xi \cap M'\xi$ is a unique M'-orbit.*

Proof: Note that $\mathcal{C}_{M'}(\theta_{x_0}) = M' \cap \mathcal{C}_G(\theta_{x_0})$ and that

$$(g \star_\theta x) x_0^{-1} = g \star_{\theta_{x_0}} (x x_0^{-1}). \tag{9}$$

The first part of the proposition now follows from (3). We proceed as in [LR03]. Suppose that $p \star x = y$, where $p \in P$ and both $x, y \in M'\xi = M'x_0$. Then

$$p \star_{\theta_{x_0}} (xx_0^{-1}) = yx_0^{-1}.$$

It follows that $p \in P \cap \theta_{x_0}(P)$. Projecting this relation to the Levi part M' of $P \cap \theta_{x_0}(P)$, we obtain for some $m \in M'$ that $m \star x = y$. ∎

Keeping the above notation, we recall the following result from [LR03].

Proposition 3.7: *Let $x \in T_0\xi \cap \mathcal{C}$. Let R be the unipotent radical of P_x, and let $\mathrm{proj}_M : P_x \to M$ be the projection onto the Levi factor M of P. Then*

(1) *the kernel of proj_M is contained in R; furthermore, proj_M maps R surjectively onto U'.*
(2) *for any function f on $P(\mathbb{A})$ which is left $U(\mathbb{A})M$-invariant, we have*

$$\int_{R\backslash R(\mathbb{A})} f(r)dr = \int_{U'\backslash U'(\mathbb{A})} f(u)du.$$

3.4 Admissible orbits. We now study the orbits $\mathcal{O}_\xi$ with $\xi \in \mathfrak{I}_M(\theta) \cap W(\theta)$. We specify the type $\mathfrak{n} = (n_1, \ldots, n_t)$ of the Levi factor M. For the rest of the subsection fix $\xi \in \mathfrak{I}_M(\theta) \cap W(\theta)$ and $x \in M\xi \cap \mathcal{C}$. Then θ_x stabilizes M and $M_x = M^{\theta_x}$. Arguing as in ([LR03], §4.3) we get that $P_x = M_x U_x$ is a Levi decomposition for P_x. Note that θ_x induces the involution $\xi\theta$ on $\mathfrak{a}_M$. From our analysis of admissible elements in $W(\theta)$, we see that there is an involution $\tau \in \mathfrak{S}_t$ associated with ξ so that

$$\xi w_0 = w_{\mathfrak{n}}(\tau; w_{n_1}, \ldots, w_{n_t}).$$

We may therefore pick a particular $x \in T_0\xi \cap \mathcal{C}$ as follows. The involution τ can be described as a product of disjoint reflections $\tau = (i_1 j_1) \cdots (i_r j_r)$. To keep our choice unique, we make the convention that $i_k < j_k, k = 1, \ldots, r$. We must have $n_{i_k} = n_{j_k}$ and, if $\tau(i) = i$, then n_i is even. We pick x so that

$$x\epsilon = w(\tau; A_1, \ldots, A_t) \tag{10}$$

where $A_{i_k} = w_{n_{i_k}}, A_{j_k} = -w_{n_{i_k}}$ and, if $\tau(i) = i$, then $A_i = \epsilon_{n_i}$. The group M_x consists of matrices of the form $\mathrm{diag}(m_1, \ldots, m_t)$ where $m_{i_k} \in GL_{n_{i_k}}, m_{j_k} = w_{n_{i_k}} {}^t m_{i_k}^{-1} w_{n_{i_k}}$ and $m_i \in H_{n_i}$ whenever $\tau(i) = i$. Thus, M_x is isomorphic to

$$GL_{n_{i_1}} \times \cdots \times GL_{n_{i_r}} \times \Big(\underset{\tau(i)=i}{\times} Sp_{n_i} \Big). \tag{11}$$

The map H_M induces isomorphisms

$$M_x(\mathbb{A})^1\backslash M_x(\mathbb{A}) \simeq (\mathfrak{a}_M)^+_{\xi\theta} \tag{12}$$

and

$$M_x(\mathbb{A})^1\backslash(M_x(\mathbb{A}) \cap G(\mathbb{A})^1) \simeq (\mathfrak{a}^G_M)^+_{\xi\theta}. \tag{13}$$

For any $x' \in M\xi \cap C$ there exists $m \in M$ such that $M_{x'} = mM_xm^{-1}$. Therefore (12) and (13) hold for $M_{x'}$ as well.

Next we quote a result of [LR03] that is used in the first reduction step for the proof of the convergence of the intertwining periods. The result is stated in [LR03] only when θ is a Galois involution. The proof, however, holds almost verbatim for our case. We therefore omit the proof. The only necessary fact is that the non-abelian cohomology $H^1(\Gamma, U)$ of a unipotent group U is trivial whenever Γ is a group of two elements of automorphisms of U.

Fix a simple root $\alpha \in \Delta_M$. Let $Q = LV$ be the parabolic subgroup of G containing P such that $\Delta^L_M = \{\alpha\}$, and let $P' = M'U'$ be the parabolic subgroup of G contained in Q with Levi factor $M' = s_\alpha M s_\alpha^{-1}$, where $s_\alpha \in W(M)$ is such that $l_M(s_\alpha) = 1$. We have $s_\alpha\alpha = -\alpha'$ where $\Delta^L_{M'} = \{\alpha'\}$. Furthermore, $U = (L \cap U)V$ and $U' = (L \cap U')V$. Let $\mathrm{proj}_L\colon Q \to L$ be the projection onto the Levi subgroup.

LEMMA 3.8: *In the above notation, assume that $-\alpha \neq \xi\theta\alpha < 0$. Set $x' = s_\alpha \star x$, and let $U^{s_\alpha}_x, P^{s_\alpha}_x$ be the conjugates of U_x, P_x, respectively, by s_α. Then we have the following.*

(1) *$U^{s_\alpha}_x = V_{x'}$; in particular, $U^{s_\alpha}_x \subset U'_{x'}$.*

(2) *The following is a short exact sequence of subgroups normalized by $M'_{x'}$:*

$$0 \to U^{s_\alpha}_x \to U'_{x'} \xrightarrow{\mathrm{proj}_L} L \cap U' \to 0.$$

(3) *If f is a function on $U'(\mathbb{A})$ which is $V(\mathbb{A})$-invariant, then*

$$\int_{U^{s_\alpha}_x(\mathbb{A})\backslash U'_{x'}(\mathbb{A})} f(u)du = \int_{L(\mathbb{A})\cap U'(\mathbb{A})} f(u)du.$$

(4) *$P^{s_\alpha}_x \subset P'_{x'}$, and a semi-invariant measure on $P^{s_\alpha}_x(\mathbb{A})\backslash P'_{x'}(\mathbb{A})$ is given by integrating over $U^{s_\alpha}_x(\mathbb{A})\backslash U'_{x'}(\mathbb{A})$.*

Through the identification (12) there is an element $\rho_x \in (\mathfrak{a}^*_M)^+_{\xi\theta}$ such that for all $m \in M_x(\mathbb{A})$ we have

$$\delta_{P_x}(m) = e^{\langle 2\rho_x, H_M(m)\rangle}.$$

In the Galois case, considered in [LR03], the convenient equality $2\rho_x = \rho_P$ holds. Unfortunately, this is not the case here. It is exactly this point that will require a slight modification of the proofs of [LR03]. The following proposition will allow us this modification.

PROPOSITION 3.9: *Let $P = MU, P' = M'U'$ be parabolic subgroups of G. Let $\xi \in \mathfrak{I}_M(\theta) \cap W(\theta)$, $\xi' \in \mathfrak{I}_{M'}(\theta) \cap W(\theta)$ and $w \in W^0(\xi, \xi')$. Let $x \in M\xi \cap C$ and denote $x' = w \star x \in M'\xi' \cap C$. Then*

$$2\rho_{x'} - \rho_{P'} = w(2\rho_x - \rho_P).$$

Proof: Assume first that $w = s_\alpha$. We may assume that $\xi\theta\alpha < 0$ (else we prove the statement for $s_\alpha^{-1} \in W^0(\xi', \xi)$). From the definition of $W^0(\xi, \xi')$ we also get that $\xi\theta\alpha \neq -\alpha$. In the proof of ([LR03], Proposition 4.3.2) it is shown that

$$\rho_{x'} = s_\alpha \rho_x + \rho_{P'}^Q$$

and that

$$2\rho_{P'}^Q = \rho_{P'} - s_\alpha \rho_P.$$

This proves the case $l_M(w) = 1$. If $l_M(w) > 1$ then it may be written as $w = s_\alpha w_1$, where $l_M(w_1) = l_M(w) - 1$, $w_1 \in W^0(\xi, w_1 * \xi)$ and $s_\alpha \in W^0(w_1 * \xi, \xi')$. The proposition follows by induction on the length $l_M(w)$. ∎

3.5 MINIMAL ADMISSIBLE ORBITS. Let P be a parabolic of G. If $x = \eta \star 1_{2n}$ is such that $\iota_M(x) \in \mathfrak{I}_M(\theta)$, we set

$$H_\eta^P = H \cap \eta^{-1} P \eta = \eta^{-1} P_x \eta.$$

Let L be a θ-stable Levi subgroup of G such that $M \subset L$. We define

$$L_\eta^P = L_H \cap \eta^{-1} P \eta.$$

The following decomposition is proved exactly as in ([LR03], Lemma 4.5.1 (3)).

LEMMA 3.10: *With notation as above we have: if $\xi \in \Xi_M(\theta) \cap W(\theta)$ and $L = L_{\xi,\theta}$ then*

$$H_\eta^P = L_\eta^P \cdot V_H.$$

LEMMA 3.11: *Let M be a parabolic of type $(n_1, \ldots, n_t), \xi \in \Xi_M(\theta) \cap W(\theta)$ and $L = L_{\xi,\theta}$. Then there exists $r \leq t/2$ such that $n_i = n_{t+1-i}$ for all $i = 1, \ldots, r$*

and $n_{r+i} = 2k_i$ is even for all $i = 1, \dots, t-2r$. If $K = k_1 + \cdots + k_{t-2r}$ then L is of type $(n_1, \dots, n_r, 2K, n_r, \dots, n_1)$.

Proof: Since L is θ-stable it is of type $(m_1, \dots, m_s) = (m_s, \dots, m_1)$, and since $M \subset L$ each of the m_i's is a sum of appropriate n_j's. Recall that $\xi = w^L_{\theta(M)}$. If $\alpha \in \Delta^L_M$ then it is associated to a pair $(j, j+1)$ of M-blocks of respective size $n_j \times n_j, n_{j+1} \times n_{j+1}$ contained in the same i-th block of size $m_i \times m_i$ of L. We claim that for any such α we must have $2i - 1 = s$, i.e. the i-th block is the central block of L. Indeed, since $\xi \in W_L$, $-\xi\theta\alpha$ 'belongs to' the $(s+1-i)$-th block of L. But since $\xi\theta\alpha = -\alpha$ we must have $s + 1 - i = i$. This shows indeed that each of the $m_i, i \neq (s-1)/2$ is a single n_j. It is only left to verify that the M-blocks in the central L-block are all even. This follows from the fact that $\xi \in W(\theta)$ by an argument we have used before, since for each such j-block, the matrix ξw_0 is a conjugate of w_0 that has w_{n_j} in a diagonal block. Hence the diagonal entries of w_{n_j} must be zero. ∎

We can now summarize. Let $\xi \in \Xi_M(\theta) \cap W(\theta)$ and $L = L_{\xi,\theta}$. By Lemma 3.11, the type of M has the form

$$(n_1, \dots, n_r, 2k_1, \dots, 2k_s, n_r, \dots, n_1),$$

and then L is of type

$$(n_1, \dots, n_r, 2K, n_r, \dots, n_1),$$

where $K = k_1 + \cdots + k_s$.

We choose $x \in T_0\xi \cap C$ as in (10). Thus, $x\epsilon$ is the matrix

$$\begin{pmatrix} & & w_N \\ & E & \\ -w_N & & \end{pmatrix} \tag{14}$$

where $N = n_1 + \cdots + n_r$ and $E = \mathrm{diag}(\epsilon_{2k_1}, \dots, \epsilon_{2k_s})$. We also make an explicit choice of $\eta \in L$ such that $\eta \star 1_{2n} = x$. We let

$$\eta = \begin{pmatrix} 1_N & & \\ & \eta_1 & \\ & & 1_N \end{pmatrix} \tag{15}$$

where η_1 is a $2K \times 2K$ permutation matrix. Using the notation of (7) with respect to the partition $\mathfrak{k} = (k_1, k_1, k_2, \dots, k_s, k_s)$, we have

$$\eta_1 = w_{\mathfrak{k}}(\tau; 1_{k_1}, 1_{k_1}, \dots, 1_{k_s}, 1_{k_s})$$

where τ is the permutation in $2s$ variables given by

$$\tau(i) = \begin{cases} 2i-1, & 1 \le i \le s, \\ 2(2s+1-i), & s+1 \le i \le 2s. \end{cases}$$

The permutation τ conjugates w_{2s} to the involution $(1,2)\cdots(2s-1,2s)$.

We finish this section with another technical lemma that we will need in order to reduce the proof of convergence of the intertwining periods to the case of minimal twisted involutions.

LEMMA 3.12: *Let $\xi \in \Xi_M(\theta) \cap W(\theta), L_{\xi,\theta} = L$ and $x \in M\xi \cap \mathcal{C}_L(\theta)$. For all $l \in L_H(\mathbb{A})$,*

$$\delta_{Q_H}(l) = e^{\langle 2\rho_x, H_L(l)\rangle}. \tag{16}$$

Proof: We let M and L be of types as given by Lemma 3.11. We first note that ρ_x is independent of the choice we make of $x \in M\xi \cap \mathcal{C}_L(\theta)$ since, by Lemma 3.6 (2), $M\xi \cap \mathcal{C}_L(\theta)$ is a unique M-orbit, and hence all P_x's are M-conjugate to each other. We thus choose x so that $x\epsilon$ is given by (14). Let $l \in L_H(\mathbb{A})$; then $l = \mathrm{diag}(g_1, \dots, g_r, h, \tilde{g}_r, \dots, \tilde{g}_1)$ where $\tilde{g}_i = w_{n_i}{}^t g_i^{-1} w_{n_i} \in GL_{n_i}(\mathbb{A})$ and $h \in H_{2K}(\mathbb{A})$. Let $l_1 = \mathrm{diag}(g_1, \dots, g_r, 1_{2K}, \tilde{g}_r, \dots, \tilde{g}_1) \in L_H(\mathbb{A})$; then $H_L(l) = H_L(l_1)$ and it is therefore enough to prove the theorem for l_1. Since $l_1 \in M_x(\mathbb{A})$, we need to show that $\delta_{P_x}(l_1) = \delta_{Q_H}(l_1)$. We can then write explicitly the conditions for a matrix in U to lie in $U_x(\mathbb{A})$ and in $V_H(\mathbb{A})$ and compare the Jacobian of the action of l_1 on each of these two unipotent groups. We leave it to the reader to verify the equality of the two Jacobians. ∎

4. Intertwining periods

We denote by $\mathcal{A}(G)$ the space of automorphic forms on $G\backslash G(\mathbb{A})$. For a parabolic P, we let $\mathcal{A}_P(G)$ be the space of automorphic forms on $U(\mathbb{A})M\backslash G(\mathbb{A})$, and we denote by $\mathcal{A}_P^1(G)$ the subspace of all $\varphi \in \mathcal{A}_P(G)$ such that for all $g \in G(\mathbb{A})$ and $a \in A_M$,

$$\varphi(ag) = e^{\langle \rho_P, H_M(a)\rangle}\varphi(g)$$

and

$$\sup_{g \in G(\mathbb{A})} |e^{-\langle \rho_P, H_M(g)\rangle}\varphi(g)| < \infty.$$

The latter condition holds whenever φ is cuspidal. The constant term along P of an automorphic form $\varphi \in \mathcal{A}(G)$ is

$$\varphi_P(g) = \int_{U\backslash U(\mathbb{A})} \varphi(ug)du.$$

For $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ we denote by $I(\lambda) = I_P(\lambda)$ the action of $G(\mathbb{A})$ on $\mathcal{A}_P(G)$ given by

$$I(\lambda, g)\varphi(g') = \varphi(g'g)e^{\langle \lambda, H_M(g'g)\rangle} e^{\langle \lambda, -H_M(g')\rangle}.$$

4.1 Definition of the intertwining periods. Let $\varphi \in \mathcal{A}^1_P(G)$ and let $\xi \in \mathfrak{I}_M(\theta) \cap W(\theta)$. Choose $x \in \mathcal{O}_\xi \cap M\xi$ and a Haar measure on $M_x(\mathbb{A})^1$. The period integral

$$P^{M_x}(\varphi)(g) = \int_{M_x \backslash M_x(\mathbb{A})^1} \varphi(mg)dm$$

is well defined. Let η be chosen so that $x = \eta \star 1_{2n}$. The intertwining period is defined by

$$J(\xi, \varphi, \lambda) = \int_{H^P_\eta(\mathbb{A}) \backslash H(\mathbb{A})} P^{M_x}(\varphi)(\eta h) e^{\langle \lambda, H_M(\eta h)\rangle} dh$$

for λ in a suitable domain of $2\rho_x - \rho_P + ((\mathfrak{a}^G_{M,\mathbb{C}})^*)^-_{\xi\theta}$ that we will specify later. To specify the quotient measure for the outer integral we recall that $H^P_\eta(\mathbb{A})$ has Levi decomposition $(\eta^{-1}M_x(\mathbb{A})\eta)(\eta^{-1}U_x(\mathbb{A})\eta)$. A measure on the vector space $M_x(\mathbb{A})^1 \backslash M_x(\mathbb{A})$ is determined by (12), and this gives a measure on $M_x(\mathbb{A})$. With this convention $J(\xi, \varphi, \lambda)$ depends on the measure on $H(\mathbb{A})$ but not on the measure on $M_x(\mathbb{A})^1$. Note that the intertwining period depends on ξ but neither on the choice of x nor η. To see that the integral makes sense formally, note that the inner period satisfies

$$P^{M_x}(\varphi)(mg) = e^{\langle \rho_P, H_M(m)\rangle} P^{M_x}(\varphi)(g)$$

for all $m \in A_M \cdot M_x(\mathbb{A})^1$ and that $M_x(\mathbb{A}) \subset A_M \cdot M_x(\mathbb{A})^1$. On the other hand, by (12) we get that

$$e^{\langle \lambda, H_M(\eta h_1 h)\rangle} = e^{\langle 2\rho_x - \rho_P, H_M(\eta h_1 \eta^{-1})\rangle} e^{\langle \lambda, H_M(\eta h)\rangle}$$

for all $h_1 \in H^P_\eta(\mathbb{A})$. So replacing h by $h_1 h$ with $h_1 \in H^P_\eta(\mathbb{A})$ changes the integrand by the factor

$$e^{\langle 2\rho_x, H_M(\eta h_1 \eta^{-1})\rangle}$$

and, by definition of ρ_x, this is exactly

$$\delta_{P_x}(\eta h_1 \eta^{-1}) = \delta_{H^P_\eta}(h_1).$$

The rest of this section and the next one will be dedicated to the convergence of $J(\xi, \varphi, \lambda)$.

4.2 Convergence statement. For each $\xi \in \mathfrak{I}_M(\theta)$ we define the cone

$$\mathcal{D}_\xi = \mathcal{D}_{\xi,M} = \{\lambda \in ((\mathfrak{a}_M^G)^*)^-_{\xi\theta} | \langle \lambda, \beta^\vee \rangle > \gamma \text{ for all } \beta \in \Phi_\xi\}$$

where $\Phi_\xi = \{\beta \in R^+(T_M, G) | \xi\theta\beta < 0\}$, and γ is a sufficiently large real number which we don't make explicit. The following result on the domains of definition of the intertwining periods is the content of [LR03; Lemma 5.2.1].

Lemma 4.1: *Let $\xi \in \mathfrak{I}_M(\theta)$.*

(1) *If $\alpha \in \Delta_M$ is such that $s_\alpha \in W(\xi, \xi'), \xi\theta\alpha < 0$, and $P' = M'U'$ is the parabolic with Levi $M' = s_\alpha M s_\alpha^{-1}$, then*

$$\mathcal{D}_{\xi,M} = s_\alpha^{-1}\mathcal{D}_{\xi',M'} \cap \{\lambda \in ((\mathfrak{a}_M^G)^*)^-_{\xi\theta} | \langle \lambda, \alpha^\vee \rangle > \gamma\}.$$

(2) *$\mathcal{D}_\xi \supset (\gamma\rho_P + ((\mathfrak{a}_M^G)^*)_+)^-_{\xi\theta}$ with equality if ξ is minimal.*

We keep the notations as in Lemma 4.1. Let $x \in M\xi \cap \mathcal{C}$ and denote $x' = s_\alpha \star x$. In light of Proposition 3.9, we have in particular that if $-\alpha \neq \xi\theta\alpha < 0$ then

$$2\rho_x - \rho_P + \mathcal{D}_\xi \subset s_\alpha^{-1}(2\rho_{x'} - \rho_{P'} + \mathcal{D}_{\xi'}). \tag{17}$$

Theorem 4.2: *Let $\varphi \in \mathcal{A}_P^1(G)$ and let $\xi \in \mathfrak{I}_M(\theta) \cap W(\theta)$. Then $J(\xi, \varphi, \lambda)$ is defined by an absolutely convergent integral when $\operatorname{Re}\lambda \in 2\rho_x - \rho_P + \mathcal{D}_\xi$. It is bounded on any set $\{\lambda | \operatorname{Re}\lambda \in D\}$ where $D \subset 2\rho_x - \rho_P + \mathcal{D}_\xi$ is compact.*

We denote by $\varphi_0 = \varphi_{0,P} \in \mathcal{A}_P^1(G)$ the function defined by

$$\varphi_0(g) = e^{\langle \rho_P, H_M(g) \rangle}.$$

We define

$$J_M(\xi, \lambda) = J_M^G(\xi, \lambda) = \int_{H_\eta^P(\mathbb{A}) \backslash H(\mathbb{A})} e^{\langle \lambda + \rho_P, H_M(\eta h) \rangle} dh.$$

Thus,

$$J_M(\xi, \lambda) = \operatorname{vol}(M_x \backslash M_x(\mathbb{A})^1)^{-1} J(\xi, \varphi_{0,P}, \lambda).$$

Theorem 4.2 is a consequence of the following.

Proposition 4.3: *Let $\xi \in \mathfrak{I}_M(\theta) \cap W(\theta)$. The integral $J_M(\xi, \lambda)$ is absolutely convergent for $\operatorname{Re}\lambda \in 2\rho_x - \rho_P + \mathcal{D}_\xi$.*

Using two reduction steps, we will reduce the proposition to the case where $\xi \in \Xi_M(\theta) \cap W(\theta)$ and $L_{\xi,\theta} = G$. We will then prove the convergence directly in this case. Denote by $S(G, M, \xi)$ the statement

$J_M(\xi,\lambda)$ *is absolutely convergent for* $\operatorname{Re}\lambda \in 2\rho_x - \rho_P + \mathcal{D}_\xi$.

Proposition 4.3 will be proved by proving the following three steps.

- *Step 1:* $S(G,M,\xi)$ for all M and all $\xi \in \Xi_M(\theta) \cap W(\theta)$ implies $S(G,M,\xi)$ for all M and all $\xi \in \mathfrak{I}_M(\theta) \cap W(\theta)$.
- *Step 2:* $S(G,M,\xi)$ for all G (i.e. for all n), for all M and for $\xi \in \Xi_M(\theta) \cap W(\theta)$ such that $L_{\xi,\theta} = G$ implies $S(G,M,\xi)$ for all G, for all M and for all $\xi \in \Xi_M(\theta) \cap W(\theta)$.
- *Step 3:* If $\xi \in \Xi_M(\theta) \cap W(\theta)$ is such that $L_{\xi,\theta} = G$ then $S(G,M,\xi)$.

In light of (17) and Lemma 3.8, step 1 is proved almost word by word as in [LR03] and we will not repeat the proof here. The proof of step 2 is again similar to that of [LR03]. We will indicate the modifications needed to take modulus functions into consideration. Later in this work we will quote without proof statements from [LR03], which require modifications of the same nature.

4.3 Proof of step 2. We now assume that $\xi \in \Xi_M(\theta) \cap W(\theta)$ and denote $L = L_{\xi,\theta}$. By Lemma 3.11 the type of M has the form

$$(n_1,\dots,n_r,2k_1,\dots,2k_s,n_r,\dots,n_1),$$

and then L is of type

$$(n_1,\dots,n_r,2K,n_r,\dots,n_1)$$

where $K = k_1 + \cdots + k_s$. We choose $x \in T_0\xi \cap C$ so that $x\epsilon$ is given by (14) and $\eta \in L$ as in (15). Let M_1 denote the Levi subgroup of GL_{2K} of type $(2k_1,\dots,2k_s)$ and P_1 the parabolic of GL_{2K} with Levi M_1. Let $\xi_1 = \eta_1 * 1_{2K}$, notation being as in (15). We define the integral

$$J^L(\xi,\varphi,\lambda) = J^L_M(\xi,\varphi,\lambda) = \int_{L^P_\eta(\mathbb{A})\backslash L_H(\mathbb{A})} P^{M_x}(\varphi)(\eta l)e^{\langle\lambda,H_M(\eta l)\rangle}\,dl. \tag{18}$$

Note that $J^L(\xi,\varphi,\lambda)$ only depends on λ^L. We also denote

$$J^L_M(\xi,\lambda) = \int_{L^P_\eta(\mathbb{A})\backslash L_H(\mathbb{A})} e^{\langle\lambda+\rho_P,H_M(\eta l)\rangle}\,dl.$$

Note that $(\mathfrak{a}^L_M)^* \simeq (\mathfrak{a}^{GL_{2K}}_{M_1})^*$ and that, identifying the two vector spaces, if $\lambda \in \mathcal{D}_{\xi,M}$ then $\lambda^L \in \mathcal{D}_{\xi_1,M_1}$. In fact it is easy to see that $(\mathcal{D}_{\xi,M})^L = \mathcal{D}_{\xi_1,M_1}$. Under this identification $\rho^Q_P = \rho_{P_1}$ and $\rho^L_x = 0 = \rho_{x_1}$ where $x_1 = \eta_1 \star 1_{2K}$. Thus, granted step 3, we have the equality

$$J^L_M(\xi,\lambda^L) = J^{GL_{2K}}_{M_1}(\xi_1,\lambda^L) \tag{19}$$

for $\lambda \in \mathcal{D}_{\xi,M}$.

LEMMA 4.4: *Let $\lambda \in 2\rho_x - \rho_P + \mathcal{D}_\xi$. Then,*

$$J(\xi, \varphi, \lambda) = J^L(\xi, e^{-\langle \rho_Q, H_L(\cdot)\rangle} \cdot \varphi^{K_H}_{|L(\mathbb{A})}, \lambda^L) \tag{20}$$

where

$$\varphi^{K_H}(g) = \int_{K_H} \varphi(gk)dk.$$

In particular,

$$J_M(\xi, \lambda) = J^L(\xi, e^{-\langle \rho_Q, H_L(\cdot)\rangle} \cdot \varphi_0^{K_H}{}_{|L(\mathbb{A})}, \lambda^L).$$

Proof: It is shown in ([LR03], Lemma 5.4.1) that

$$J(\xi, \varphi, \lambda) =$$
$$\int_{Q_H(\mathbb{A})\backslash H(\mathbb{A})} \int_{L_\eta^P(\mathbb{A})V(\mathbb{A})\backslash Q_H(\mathbb{A})} \delta_{Q_H}(q)^{-1} e^{\langle \lambda, H_M(\eta q h)\rangle} P^{M_x}(\varphi)(\eta q h) dq dh.$$

By (16) and Lemma 3.10, this is equal to

$$\int_{K_H} \int_{L_\eta^P(\mathbb{A})\backslash L_H(\mathbb{A})} e^{-\langle 2\rho_x, H_L(l)\rangle} e^{\langle \lambda, H_M(\eta l k)\rangle} P^{M_x}(\varphi)(\eta l k) dl dk. \tag{21}$$

Note that

$$\begin{aligned}\langle \lambda, H_M(\eta l)\rangle &= \langle \lambda^L, H_M(\eta l)\rangle + \langle \lambda_L, H_L(\eta l)\rangle \\ &= \langle \lambda^L, H_M(\eta l)\rangle + \langle \lambda_L, H_L(l)\rangle = \langle \lambda^L, H_M(\eta l)\rangle + \langle 2\rho_x - \rho_Q, H_L(l)\rangle.\end{aligned}$$

The last equality is explained as follows. By (6), $\rho_x \in \mathfrak{a}_L^*$, therefore $(2\rho_x - \rho_P)_L = 2\rho_x - \rho_Q$. By (5), $(\mathcal{D}_\xi)_L \subset (\mathfrak{a}_L^*)_\theta^-$, and for $l \in L_H(\mathbb{A})$ we have $H_L(l) \in (\mathfrak{a}_L)_\theta^+$. We can therefore conclude, as in [LR03], that (21) is equal to

$$\int_{L_\eta^P(\mathbb{A})\backslash L_H(\mathbb{A})} P^{M_x}(e^{-\langle \rho_Q, H_L(\cdot)\rangle} \varphi^{K_H})(\eta l) e^{\langle \lambda^L, H_M(\eta l)\rangle} dl,$$

and the lemma now follows. ∎

It is left to note that $J^L(\xi, e^{-\langle \rho_Q, H_L(\cdot)\rangle} \cdot \varphi_0^{K_H}{}_{|L(\mathbb{A})}, \lambda^L)$ is bounded above by a constant multiple of $J_M^L(\xi, \lambda^L)$. Thanks to (19), step 2 now follows. ∎

5. Proof of step 3

Assume now that M is of type $\mathfrak{K} = (2k_1, \dots, 2k_s)$ and $n = k_1 + \cdots + k_s$. Let $\xi \in \Xi_M(\theta) \cap W(\theta)$ be the unique twisted involution such that $L_{\xi,\theta} = G$. In the notation of (7)

$$\xi = w_{\mathfrak{K}}(w_s; I_{2k_1}, \dots, I_{2k_s}).$$

With our favorite choice of $x = \eta \star 1_{2K}$ given by (14) and (15) we have $\rho_x = 0$. By Lemma 4.1 we then see that for $\lambda \in 2\rho_x - \rho_P + \mathcal{D}_\xi$, $\lambda + \rho_P$ lies in $\gamma\rho_P + (\mathfrak{a}_M^*)_+$. For such λ we will prove the convergence of the intertwining period

$$J(\xi, \lambda) = \int_{P_x(\mathbb{A})\backslash H_x(\mathbb{A})} e^{\langle \lambda, H_M(h\eta) \rangle} dh$$

if γ is large enough. We denote the matrix $x\epsilon$ of (14) by $E_{\mathfrak{K}}$. To prove the convergence of the integral we will use a convenient system of coordinates for $P_x(\mathbb{A})\backslash H_x(\mathbb{A})$. This was done in [JR92b] when $s = 2$ and $k_1 = k_2$. We extend the integration formula of Jacquet and Rallis to any partition. We first set up the notations. Let $H_{\mathfrak{K}}$ be the symplectic group in G obtained from the skew-symmetric form defined by the matrix $E_{\mathfrak{K}}$. Let $T = H_{2k_1} \times \cdots \times H_{2k_s}$ imbedded in $H_{\mathfrak{K}}$ in diagonal blocks. Then with the above notation, $H_x = \eta H \eta^{-1} = H_{\mathfrak{K}}$ and $T = P_x$. We describe certain parabolic subgroups of $H_{\mathfrak{K}}$. Let $M_{\mathfrak{K}}$ be the subgroup of $H_{\mathfrak{K}}$ consisting of matrices of the form

$$\mathrm{diag}(g_1, \tilde{g}_1, \dots, g_{s-1}, \tilde{g}_{s-1}, h) \tag{22}$$

with $g_i \in GL_{k_i}$, $\tilde{g}_i = w_{k_i}{}^t g^{-1} w_{k_i}$ and $h \in H_{2k_s}$. We also define unipotent groups by recursion. For an integer k we define $U_k = \{1_k\}$ the trivial group. If $s > 1$, then for the partition $\mathfrak{K}$ we let $U_{\mathfrak{K}}$ be the subgroup of $H_{\mathfrak{K}}$ of matrices of the form

$$\begin{pmatrix} 1_{k_1} & Z & Y \\ 0 & 1_{k_1} & 0 \\ 0 & X & u \end{pmatrix} \tag{23}$$

where, denoting by $\mathfrak{K}^{(1)}$ the partition $(2k_2, \dots, 2k_s)$, we have that Y is a $k_1 \times 2(k_2 + \cdots + k_s)$ matrix satisfying

$$Y = w_{k_1}{}^t X^t u^{-1} E_{\mathfrak{K}^{(1)}},$$

Z is a $k_1 \times k_1$ matrix satisfying

$${}^t Z w_{k_1} - w_{k_1} Z + {}^t X^t u^{-1} E_{\mathfrak{K}^{(1)}} u^{-1} X = 0$$

and

$$u \in U_{\mathfrak{K}(1)}.$$

An element of the form (23) will be denoted

$$v(X, Z, u).$$

We have that $Q_{\mathfrak{K}} = M_{\mathfrak{K}} U_{\mathfrak{K}}$ is a Levi decomposition of a parabolic subgroup of $H_{\mathfrak{K}}$. Note that $M_{\mathfrak{K}} \subset T$ and that $Q_{\mathfrak{K}} \cap T$ is a parabolic subgroup of T with Levi decomposition $Q_{\mathfrak{K}} \cap T = M_{\mathfrak{K}} V$, where $V = T \cap U_{\mathfrak{K}}$. Any element h of $H_{\mathfrak{K}}$ can therefore be written (non-uniquely) in the form $h = tuk$ with $t \in T, u \in U_{\mathfrak{K}}$ and $k \in K \cap H_{\mathfrak{K}}$. We introduce a section of $V \backslash U_{\mathfrak{K}}$. First, if $n < N$ let $U_{\mathfrak{K}}^N$ be the group $U_{\mathfrak{K}}$ imbedded in GL_{2N} in the bottom right $2n \times 2n$-block, i.e. it is the group of matrices of the form

$$\mathrm{diag}(1_{2(N-n)}, u)$$

with $u \in U_{\mathfrak{K}}$. If $s = 2$ for $X \in M_{2k_2 \times k_1}$ let

$$\sigma_{k_1,k_2}(X) = \sigma(X) = v(X, \frac{1}{2} w_{k_1} {}^t X \epsilon_{2k_2} X, 0).$$

For $s > 2$ we then define

$$\sigma(X_1, \dots, X_{s-1}) = \sigma^n_{k_{s-1},k_s}(X_{s-1}) \cdots \sigma^n_{k_2,n-k_1-k_2}(X_2) \sigma_{k_1,n-k_1}(X_1)$$

where $\sigma^n_{k_i,k_{i+1}+\dots+k_s}(X)$ denotes the imbedding of $\sigma_{k_i,k_{i+1}+\dots+k_s}(X) \in U_{(2k_i,2k_{i+1},\dots,2k_s)}$ into $U^n_{(2k_i,2k_{i+1},\dots,2k_s)} \subset U_{\mathfrak{K}}$. Then the map

$$X_1, \dots, X_{s-1} \mapsto \sigma(X_1, \dots, X_{s-1})$$

from $M_{2(K-k_1) \times k_1} \times \dots \times M_{2k_s \times k_{s-1}}$ to $U_{\mathfrak{K}}$ defines a bijection

$$M_{2(n-k_1) \times k_1} \times M_{2(k_3+\dots k_s) \times k_2} \times \dots \times M_{2k_s \times k_{s-1}} \simeq V \backslash U_{\mathfrak{K}}.$$

5.1 The local integration formula. We now assume that F is a local field. We define $\|X\|$ and $\lambda(X)$ as in [JR92b]. If F is non-archimedean, then for any matrix X we will denote by $\|X\|$ the supremum of the absolute values of the entries of X and we set

$$\lambda(X) = \max(1, \|X\|).$$

If F is real we let $\|X\|^2$ be the sum of squares of the entries of X and set

$$\lambda(X) = \sqrt{1 + \|X\|^2},$$

and if F is complex we let $\|X\|^2$ be the sum of products of the entries of X with their complex conjugates and set

$$\lambda(X) = 1 + \|X\|^2.$$

Let Φ be the function on $H_{\mathfrak{K}}$ defined by

$$\Phi(h) = \delta_{Q\cap T}(m)$$

when $h = muk$, with $m \in M_{\mathfrak{K}}, u \in U_{\mathfrak{K}}$ and $k \in K \cap H_{\mathfrak{K}}$. If $m \in M_{\mathfrak{K}}$ is given by (22), then

$$\Phi(m) = |\det g_1|^{k_1+1} \cdots |\det g_{s-1}|^{k_{s-1}+1}.$$

For all $h \in H_{\mathfrak{K}}$ the function $t \mapsto \Phi(th)$ is $(Q \cap T, \delta_{Q\cap T})$-equivariant. Therefore, integrating over $K_T = K \cap T$ provides a left T-invariant function

$$\Phi_1(h) = \int_{K_T} \Phi(k_T h) dk_T.$$

Thus Φ_1 is a positive continuous function on $H_{\mathfrak{K}}$, which is left T-invariant. We now set

$$\gamma(h) = \Phi_1(h)^{-1}$$

and provide a Haar measure on $H_{\mathfrak{K}}$ in terms of $T \times (V\backslash U_{\mathfrak{K}}) \times (K \cap H_{\mathfrak{K}})$ that generalizes that of [JR92b]. The proof is exactly as in [JR92b] and we omit it.

Proposition 5.1: *For a continuous function of compact support on $H_{\mathfrak{K}}$, the integral*

$$\int f(tuk) dt \gamma(u) du dk$$

converges absolutely and defines a Haar measure on $H_{\mathfrak{K}}$. Here, dt is a Haar measure on T, dk a Haar measure on $K \cap H_{\mathfrak{K}}$ and du a $U_{\mathfrak{K}}$-invariant measure on $V\backslash U_{\mathfrak{K}}$.

We set $\gamma(X_1, \ldots, X_{s-1}) = \gamma(\sigma(X_1, \ldots, X_{s-1}))$. We can express the $U_{\mathfrak{K}}$-invariant measure on $V\backslash U_{\mathfrak{K}}$ in terms of the section σ. Thus the integral

$$\int f(t\sigma(X_1, \ldots, X_{s-1})k) dt \gamma(X_1, \ldots, X_{s-1}) dX_{s-1} \cdots dX_1 dk \tag{24}$$

defines a Haar measure on $H_{\mathfrak{K}}$. In order to obtain a similar global integration formula, we need a majorization of γ.

PROPOSITION 5.2: *There is a positive constant c and a positive integer m such that*

$$\gamma(X_1,\dots,X_{s-1}) \le c\Big(\prod_{i=1}^{s-1}\lambda(X_i)\Big)^m.$$

The integer m is dependent on the partition $\mathfrak{K}$ but not on the field F. Assume that F is non-archimedean of odd residual characteristic. Then we can take $c = 1$ and, furthermore, if all X_i's have integral entries then $\gamma(X_1,\dots,X_{s-1}) = 1$.

Proof: Let e_i be the canonical basis of the space of $2n$-dimensional row vectors. Set

$$\alpha_i = e_{n_i+1} \wedge \cdots \wedge e_{2n}$$

where $n_i = 2(k_1 + \cdots + k_{i-1}) + k_i$. For any $h \in H_{\mathfrak{K}}$,

$$\Phi(h) = \prod_{i=1}^{s-1} \|\alpha_i h\|^{-k_i-1}.$$

Therefore,

$$\Phi(h) \ge \Big(\prod_{i=1}^{s-1} \|\alpha_i\|^{-k_i-1}\Big)\prod_{i=1}^{s-1}\|h\|_i^{-k_i-1}$$

where $\|h\|_i$ is the norm of h in the appropriate exterior power. Integrating over K_T we get the same lower bound for Φ_1 and we therefore get that

$$\gamma(h) \le \Big(\prod_{i=1}^{s-1} \|\alpha_i\|^{k_i+1}\Big)\prod_{i=1}^{s-1}\|h\|_i^{k_i+1}.$$

Since the absolute value of each entry of $\sigma(X_1,\dots,X_{s-1})$ is majorized by some power of $\prod_{i=1}^{s-1}\lambda(X_i)$, the power being independent of F, it is clear that the same holds for each of the (compatible) norms $\|\cdot\|_i$ applied to $\sigma(X_1,\dots,X_{s-1})$. The proposition readily follows. ∎

COROLLARY 5.3: *Assume that F is non-archimedean of odd residual characteristic and our choices of Haar measures are normalized so that $K \cap H_{\mathfrak{K}}$ has volume 1 for dk, K_T has volume 1 for dt and the set of integral matrices has volume 1 for dX_i. Then in the integration formula (24) we obtain the Haar measure dh on $H_{\mathfrak{K}}$ that gives volume 1 to $K \cap H_{\mathfrak{K}}$.*

5.2 THE GLOBAL INTEGRATION FORMULA. Let F be a number field now. We define the global expressions $\|X\|, \lambda(X)$ and $\gamma(X_1,\dots,X_{s-1})$ as the product over all places of F of the corresponding local expressions. We conclude from the local formula a global integration formula.

PROPOSITION 5.4: *The integral*

$$\int f(t\sigma(X_1,\dots,X_{s-1})k)dt\gamma(X_1,\dots,X_{s-1})dX_{s-1}\cdots dX_1dk$$

defines a Haar measure on $H_{\mathfrak{K}}(\mathbb{A})$. There is a positive constant c and a positive integer m such that

$$\gamma(X_1,\dots,X_{s-1})\le c\Big(\prod_{i=1}^{s-1}\lambda(X_i)\Big)^m. \tag{25}$$

5.3 THE CONVERGENCE. We denote by $P=MU$ the Levi decomposition of the standard parabolic of G of type $\mathfrak{K}$. We can identify $\mathfrak{a}_M$ with $\mathbb{R}^s$. For $\lambda=(\lambda_1,\dots,\lambda_s)\in\mathbb{R}^s$ and $g=umk\in G(\mathbb{A})$, where $u\in U(\mathbb{A}), m\in M(\mathbb{A}), k\in K$, we can then write

$$e^{\langle\lambda,H_M(g)\rangle}=|\det m_1|^{\lambda_1}\cdots|\det m_s|^{\lambda_s}$$

where m_i is the $2k_i\times 2k_i$ diagonal block of m. Let $e_i, i=1,\dots,2n$ be the canonical basis of the space of $2n$-dimensional row vectors. Let $\epsilon_i=e_{2(k_1+\cdots+k_{i-1})+1}\wedge\cdots\wedge e_{2K}, i=2,3,\dots,s$. Then for g as above

$$\|\epsilon_i g\|=|\det m_i||\det m_{i+1}|\cdots|\det m_s|.$$

Therefore, for $g\in G(\mathbb{A})^1$ we have

$$e^{\langle\lambda,H_M(g)\rangle}=\prod_{i=2}^{s}\|\epsilon_i g\|^{-(\lambda_{i-1}-\lambda_i)}. \tag{26}$$

LEMMA 5.5: *For all $i=2,\dots,s$ we have*

$$\|\epsilon_i\sigma(X_1,\dots,X_{s-1})\|\ge\lambda(X_{i-1}).$$

Proof: Note that $\epsilon_i g$ has as coordinates the $2(k_i+\cdots+k_s)\times 2(k_i+\cdots+k_s)$ minors of the bottom $2(k_i+\cdots+k_s)$ rows of g. From the definition of $\sigma(X_1,\dots,X_{s-1})$ we get that its bottom $2(k_i+\cdots+k_s)$ rows contain the $2(k_i+\cdots+k_s)\times k_{i-1}$-block

$$\sigma(X_{s-1},\dots,X_i)X_{i-1}$$

and the block

$$\sigma(X_{s-1},\dots,X_i).$$

Since $\det\sigma(X_{s-1},\dots,X_i)=1$, multiplying by $\sigma(X_{s-1},\dots,X_i)^{-1}$ from the left we see that the $2(k_i+\cdots+k_s)\times 2(k_i+\cdots+k_s)$ minors of

$$(X_{i-1},1_{2(k_i+\cdots+k_s)})$$

are coordinates of $\epsilon_i \sigma(X_1, \ldots, X_{s-1})$. Since, in particular, each entry of X_{i-1} can be obtained as such a minor, the lemma follows. ∎

To prove step 3 we need to prove that if $\lambda = (\lambda_1, \ldots, \lambda_s)$ with

$$\lambda_i > \gamma + \lambda_{i+1},$$

then for γ large enough the integral

$$\int_{T(\mathbb{A})\backslash H_{\mathfrak{R}}(\mathbb{A})} e^{\langle \lambda, H_M(h\eta)\rangle} dh$$

converges. Since $\eta \in K$ we can omit it, and using the integration formula of Jacquet and Rallis this integral becomes

$$\int e^{\langle \lambda, H_M(\sigma(X_1,\ldots,X_{s-1}))\rangle} \gamma(X_1, \ldots, X_{s-1}) dX_{s-1} \cdots dX_1.$$

By Lemma 5.5, formula (26) and the majorization (25), the convergence will follow from the convergence of

$$\prod_{i=2}^{s} \int \lambda(X_{i-1})^{m-(\lambda_{i-1}-\lambda_i)} dX_i.$$

For $\gamma >> m$ this is proved exactly as in ([JR92b], Proposition 7). ∎

6. Periods of pseudo-Eisenstein series

Fix a Levi subgroup M. Let $\mathcal{A}^1(G)_c$ be the space of cusp forms in $\mathcal{A}(G)$ which are invariant under A_0. From [JR92b], we have the following result of Jacquet and Rallis.

PROPOSITION 6.1: *Let $\varphi \in \mathcal{A}^1(G)_c$. Then for any $g \in G(\mathbb{A})$,*

$$\int_{H\backslash H(\mathbb{A})} \varphi(hg) dh = 0.$$

Remark: We note that if φ is a cusp form on G that satisfies

$$\varphi(ag) = e^{\langle \mu, H_G(a)\rangle} \varphi(g)$$

for $a \in A_0$, then the proposition of Jaquet and Rallis still holds. Indeed, the function $\varphi_1(g) = e^{-\langle \mu, H_G(g)\rangle} \varphi(g)$ is in $\mathcal{A}^1(G)_c$ and the symplectic periods of φ and of φ_1 coincide.

We will also denote by $\mathcal{A}^1_P(G)_c$ the space of cusp forms in $\mathcal{A}^1_P(G)$. For $\varphi \in \mathcal{A}^1_P(G)_c$, we define the Eisenstein series $E(\varphi,\lambda)$ as the analytic continuation of

$$E(g,\varphi,\lambda) = \sum_{\delta \in P\backslash G} \varphi(\delta g) e^{\langle \lambda, H_M(\delta g)\rangle}$$

to $\lambda \in (\mathfrak{a}^G_{M,\mathbb{C}})^*$. The series converges absolutely if $\lambda - \rho_P \in (\mathfrak{a}^G_M)^*_+$ and defines an automorphic form in $\mathcal{A}(G)$. For any $w \in W(M,M')$ with $P' = M'U'$ the parabolic associated to the Levi M', the intertwining operator $M(w,\lambda)$ is defined by

$$M(w,\lambda)\varphi(g) = e^{-\langle w\lambda, H_0(g)\rangle} \int_{(U'(\mathbb{A})\cap wU(\mathbb{A})w^{-1})\backslash U'(\mathbb{A})} \varphi(w^{-1}ug) e^{\langle \lambda, H_0(w^{-1}ug)\rangle} du.$$

Its domain of convergence includes that of the Eisenstein series.

Let $\mathcal{P}((\mathfrak{a}^G_{M,\mathbb{C}})^*)$ be the Paley–Wiener space of functions on $(\mathfrak{a}^G_{M,\mathbb{C}})^*$ obtained as Fourier transforms of compactly supported smooth functions on $\mathfrak{a}^G_M$. For a finite-dimensional subspace $\mathcal{V}$ of $\mathcal{A}^1_P(G)_c$, let $\mathcal{P}_{(M,\mathcal{V})}$ be the space of $\mathcal{V}$-valued holomorphic and Paley–Wiener functions on $(\mathfrak{a}^G_{M,\mathbb{C}})^*$. We may identify $\mathcal{P}_{(M,\mathcal{V})}$ with $\mathcal{P}((\mathfrak{a}^G_{M,\mathbb{C}})^*) \otimes \mathcal{V}$. For any $\phi \in \mathcal{P}_{(M,\mathcal{V})}$, we define the continuous function F_ϕ on $U(\mathbb{A})M\backslash G(\mathbb{A})$ by

$$F_\phi(g) = \int_{i(\mathfrak{a}^G_M)^*} \phi(\lambda)(g) d\lambda$$

and the pseudo-Eisenstein series

$$\theta_\phi(g) = \sum_{\gamma \in P\backslash G} F_\phi(\gamma g).$$

By [MW94], the sum is absolutely convergent, θ_ϕ is rapidly decreasing, and we have

$$\theta_\phi(g) = \int_{\operatorname{Re}\lambda = \lambda_0} E(g,\phi(\lambda),\lambda) d\lambda$$

for any λ_0 in the region of convergence of the Eisenstein series.

6.1 Some obvious vanishing. Let $\xi \in \mathfrak{J}_M(\theta)$, and denote by $\mathfrak{n} = (n_1,\ldots,n_t)$ the type of the Levi M. Our analysis of admissible orbits implies that there is an involution $\tau \in \mathfrak{S}_t$ such that $\xi w_0 = w_{\mathfrak{n}}(\tau; w_{n_1},\ldots,w_{n_t})$. We define the set

$$W_M(\theta) = \{\xi \in \mathfrak{J}_M(\theta) | \tau_\xi \text{ has no fixed points }\}. \tag{27}$$

Thus, $W_M(\theta)$ is empty unless t is even and there is a θ-stable Levi in the associate class of M. In any case it is clear that $W_M(\theta) \subset W(\theta)$. We remark that the elements of $W_M(\theta)$ are exactly those $\xi \in \mathfrak{I}_M(\theta) \cap W(\theta)$ such that $W(\xi, 1)$ is not empty.

PROPOSITION 6.2: *Let $\varphi \in \mathcal{A}^1_P(G)_c$ and $\xi \in \mathfrak{I}_M(\theta) \cap W(\theta)$ such that $\xi \notin W_M(\theta)$. For λ in the domain of convergence we have*

$$J(\xi, \varphi, \lambda) = 0.$$

Proof: Choosing $x \in T_0\xi \cap C$ as in (10), we see from (11) that the inner period integral $P^{M_x}(\varphi)$ will involve a symplectic period of a cusp form on a certain block GL_{2k} of M. By the remark following Proposition 6.1, we conclude that the inner period vanishes. ∎

6.2 DISTRIBUTIONAL FORMULA FOR THE PERIOD.

THEOREM 6.3: *For each $\xi \in W_M(\theta)$, let $x \in T_0\xi \cap C$ be chosen as in (10) and choose an element $\lambda_0(x) \in 2\rho_x - \rho_P + \mathcal{D}_\xi$. Then*

$$\int_{H\backslash H(\mathbb{A})} \theta_\phi(h)dh = \sum_{\xi \in W_M(\theta)} \int_{\lambda_0(x)+i((\mathfrak{a}^G_M)^*)^-_{\xi\theta}} J(\xi, \phi(\lambda), \lambda)d\lambda. \tag{28}$$

Proof: The proof is almost identical to that of ([LR03], Theorem 7.1.1). Since the series $\sum_{\gamma\in P\backslash G} |F_\phi(\gamma g)|$ is rapidly decreasing, it is in particular integrable over $H\backslash H(\mathbb{A})$. We can therefore write

$$\int_{H\backslash H(\mathbb{A})} \theta_\phi(h)dh = \sum_\eta \int_{H^P_\eta\backslash H(\mathbb{A})} F_\phi(\eta h)dh$$

where the sum ranges over the set $\{\eta\}$ of double coset representatives for $P\backslash G/H$. Let $x = \eta \star 1_{2n}$. By Proposition 3.5, for each η there is associated a unique $\xi \in {}_MW_{\theta(M)} \cap W(\theta)$ so that $\iota_M(x) = \xi$. As in [LR03], we use Proposition 3.7 to show that if ξ is not admissible, then the summand associated with it vanishes. We are therefore only left with a sum over η so that the associated ξ is admissible. Proceeding as in [LR03], we may write

$$\int_{H^P_\eta\backslash H(\mathbb{A})} F_\phi(\eta h)dh =$$

$$\int_{H^P_\eta(\mathbb{A})\backslash H(\mathbb{A})} \int_{M_x\backslash M_x(\mathbb{A})^1} \int_{(\mathfrak{a}^G_M)^+_{\xi\theta}} e^{-\langle 2\rho_x, \nu\rangle} F_\phi(e^\nu m\eta h)d\nu dm dh. \tag{29}$$

From [MW94] we get that for any $\lambda_0 \in (\mathfrak{a}^G_M)^*$,

$$F_\phi(g) = \int_{\lambda_0+i(\mathfrak{a}^G_M)^*} \phi(\lambda)(g)d\lambda,$$

and the inversion formula for the Fourier transform gives

$$\phi(\lambda)(g) = \int_{\mathfrak{a}_M^G} F_\phi(e^x g) e^{-\langle \lambda+\rho_P, x+H_M(g)\rangle} dx.$$

Applying partial Fourier inversion to (29) we get that for any $\lambda_0 \in ((\mathfrak{a}_M^G)^*)^-_{\xi\theta}$, (29) equals

$$\int_{H_\eta^P(\mathbb{A})\backslash H(\mathbb{A})} \int_{M_x\backslash M_x(\mathbb{A})^1} \int_{\lambda_0+i((\mathfrak{a}_M^G)^+_{\xi\theta})^\perp} \phi(2\rho_x - \rho_P + \lambda)(m\eta h) e^{\langle 2\rho_x - \rho_P+\lambda, H(\eta h)\rangle} d\nu dm dh.$$

The same argument as in [LR03] implies now that if $\lambda_0 \in \mathcal{D}_\xi$, then we can interchange the inner integral with the outer integrals to obtain

$$\int_{\lambda_0+i((\mathfrak{a}_M^G)^-_{\xi\theta})} J(\xi, \phi(\lambda), \lambda) d\lambda.$$

The theorem now follows from Proposition 6.2. ∎

7. The period of a truncated Eisenstein series

Our next goal is to obtain a formula, analogous to Theorem 11.1.1 in [LR03], of the period of a truncated Eisenstein series. We will follow the argument there closely. Since it is of an inductive nature, we will need to prove it for θ-stable Levi subgroups of GL_{2n}. It will therefore be convenient to change notation until we prove Theorem 7.5. In §7.4 we will go back to our original notation. Fix a Levi subgroup of GL_{2n} of type $(n_1, \ldots, n_r, 2K, n_r, \ldots, n_1)$. We allow the case $K = 0$. Until further notice we will denote this Levi subgroup by G. This is the reason why most of the notation in §2 was set up for such a G. Thus, H is the group of θ-fixed points in G. It is the intersection of G with the symplectic group $H_{2n} = Sp_{2n}$. The spaces $\mathcal{A}(G), \mathcal{A}_P(G)$ and $\mathcal{A}_P^1(G)$ for a parabolic subgroup P of G, of automorphic forms, are defined for G in a way similar to our definitions for GL_{2n}. By ([MW94], §1.3.2), a function $\varphi \in \mathcal{A}_P(G)$ admits a decomposition

$$\varphi(umk) = \sum_i Q_i(H_M(m))\psi_i(mk)$$

where $Q_i \in \mathbb{C}[\mathfrak{a}_M]$, and $\psi_i \in \mathcal{A}_P(G)$ satisfies

$$\psi_i(ag) = e^{\langle \lambda_i+\rho_P, H_M(a)\rangle}\psi_i(g)$$

for $a \in A_M$. The $\lambda_i \in \mathfrak{a}^*_{M,\mathbb{C}}$ are uniquely determined and are called the exponents of φ. For $\varphi \in \mathcal{A}_P(G)$ and $Q \subset P$ the exponents of φ along Q are defined to be the exponents of φ_Q. We denote them by $\mathcal{E}_Q(\varphi)$. We then denote

$$\mathcal{E}(\varphi) = \bigcup_{Q\subset P} \mathcal{E}_Q(\varphi).$$

7.1 Mixed truncation. The map $P \mapsto P_H = P \cap H$ is a one to one correspondence between θ-stable parabolic subgroups of G and parabolic subgroups of H. As in [JLR99] and [LR03], it will be convenient to use the mixed truncation of a function φ on $G\backslash G(\mathbb{A})$. For any parabolic subgroups $P \subset Q$ of G, let τ_P^Q be the characteristic function of

$$\{X \in \mathfrak{a}_0 | \langle \alpha, X\rangle > 0 \text{ for all } \alpha \in \Delta_P^Q\}$$

and $\hat{\tau}_P^Q$ be the characteristic function of

$$\{X \in \mathfrak{a}_0 | \langle \omega, X\rangle > 0 \text{ for all } \omega \in \hat{\Delta}_P^Q\}.$$

For any $X, H \in \mathfrak{a}_P$, let

$$\Gamma_P(H,X) = \sum_{P\subset Q} (-1)^{\dim \mathfrak{a}_Q^G} \tau_P^Q(H)\hat{\tau}_Q(H-X).$$

This is a compactly supported function, defined by Arthur in [Art81] (and denoted there with a prime). Since the spaces $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$ are the same for G as they are for GL_{2n}, θ acts on them as the involution (4). The projections into the ± 1-eigenspaces of θ have therefore been defined. Let $\rho_{P_H} \in (\mathfrak{a}_P^*)_\theta^+$ be so that

$$\delta_{P_H}(\cdot) = e^{\langle 2\rho_{P_H}, H_P(\cdot)\rangle}.$$

The mixed truncation is defined for $T \in (\mathfrak{a}_0)_\theta^+$ sufficiently positive by

$$\Lambda_m^T\varphi(h) = \sum_{P_H\subset H} (-1)^{\dim((\mathfrak{a}_P)_\theta^+)} \sum_{\delta\in P_H\backslash H} \varphi_P(\delta h)\hat{\tau}_P(H_P(\delta h) - T).$$

Similarly for a θ-stable parabolic Q, we define $\Lambda_m^{T,Q}$ by

$$\Lambda_m^{T,Q}\varphi(h) = \sum_{P_H\subset Q_H} (-1)^{\dim((\mathfrak{a}_P^Q)_\theta^+)} \sum_{\delta\in P_H\backslash Q_H} \varphi_P(\delta h)\hat{\tau}_P^Q(H_P(\delta h) - T).$$

The mixed truncation satisfies properties analogous to Arthur's truncation operator Λ^T. In the Galois case these properties are proved in [LR03]. Their proof is valid word by word for our case; we therefore only state the result.

LEMMA 7.1: *Let $\varphi \in \mathcal{A}(G)$. Then*

(1) $\Lambda_m^T \varphi$ *is rapidly decreasing on* $H\backslash H(\mathbb{A})^1$;

(2) *we have*

$$\varphi(h) = \sum_{P_H \subset H} \sum_{P_H\backslash H} \Lambda_m^{T,P}\varphi(\delta h)\tau_P(H_P(\delta h) - T); \tag{30}$$

(3) *also*

$$\Lambda_m^{T+T'}\varphi(h) = \sum_{P_H \subset H} \sum_{P_H\backslash H} \Lambda_m^{T,P}\varphi(\delta h)\Gamma_P(H_P(\delta h) - T, T'). \tag{31}$$

7.2 THE REGULARIZED PERIOD INTEGRAL. The regularization of the period integral in [JLR99] and in [LR03] is based on a regularization of integrals of exponential polynomial functions over cones in vector spaces. A detailed discussion concerning exponential polynomials and the regularized integrals involved is provided in ([JLR99], §1). To apply the regularization to the symplectic periods case, we modify the definitions of some spaces of automorphic forms from [LR03] to take the modulus functions into account. We will quote results from [LR03] without proof. The only modification required to validate them in our case is in the nature explained in our proof of Lemma 4.4. We define the regularized period integral on the space $\mathcal{A}(G)'$ of automorphic forms φ for which for all parabolic subgroups P_H of H, $\lambda \in \mathcal{E}_P(\varphi)$ an exponent of φ along P and $\varpi \in \hat{\Delta}^H_{P_H}$, we have

$$\langle \lambda, \varpi^\vee \rangle \neq \langle 2\rho_{P_H} - \rho_P, \varpi^\vee \rangle.$$

For $\varphi \in \mathcal{A}(G)'$, we define

$$\int^*_{H\backslash H(\mathbb{A})} \varphi(h)dh = \sum_{P_H} \int^{\#}_{P_H\backslash H(\mathbb{A})} \Lambda_m^{T,P}\varphi(h)\tau_{P_H}(H_P(h) - T)dh, \tag{32}$$

where

$$\int^{\#}_{P_H\backslash H(\mathbb{A})} \Lambda_m^{T,P}\varphi(h)\tau_{P_H}(H_P(h) - T)dh =$$

$$\int_{K_H}\int_{M_H\backslash M_H(\mathbb{A})^1}[\int^{\#}_{(\mathfrak{a}_P)^+_\theta} \Lambda_m^{T,P}\varphi(e^X mk)e^{-\langle 2\rho_{P_H}, X\rangle}\tau_{P_H}(X-T)dX]dmdk$$

and the $\#$-integral of a polynomial exponential function over a cone in a vector space is defined in [JLR99]. The following result summarizes the properties of the regularized period. It is Theorem 8.4.1 in [LR03].

THEOREM 7.2:

- *The regularized integral is well defined and depends only on the choice of Haar measures. It is independent of T and the choice of P_0 and K.*
- *The map $\varphi \mapsto \int^*_{H\backslash H(\mathbb{A})} \varphi(h)dh$ is a right-$H(\mathbb{A})$-invariant functional on $\mathcal{A}(G)'$.*
- *If $\varphi \in \mathcal{A}(G)$ is integrable over $H\backslash H(\mathbb{A})$, then $\varphi \in \mathcal{A}(G)'$ and*

$$\int^*_{H\backslash H(\mathbb{A})} \varphi(h)dh = \int_{H\backslash H(\mathbb{A})} \varphi(h)dh.$$

- *Let φ_λ be an analytic family of automorphic forms, and let $\mathcal{O}$ be the set of all λ such that $\varphi_\lambda \in \mathcal{A}(G)'$. Then $\mathcal{O}$ is an open set and $\lambda \mapsto \int^*_{H\backslash H(\mathbb{A})} \varphi_\lambda(h)dh$ is analytic on $\mathcal{O}$.*

Another characterization of the regularized period is given in Proposition 8.4.1 in [LR03].

PROPOSITION 7.3:

- (1) *For any $\varphi \in \mathcal{A}(G)$, the function $T \mapsto \Lambda^T_m\varphi(h)dh$ equals a polynomial exponential $\sum p_\lambda(T)e^{\langle\lambda,T\rangle}$ for $T \in (\mathfrak{a}_0)^+_\theta$ sufficiently positive. The exponents may be taken from the set*

$$\bigcup_{P_H}(\rho_P - 2\rho_{P_H} + \mathcal{E}_P(\varphi)).$$

- (2) *If $\varphi \in \mathcal{A}(G)'$, then*

$$\int^*_{H\backslash H(\mathbb{A})} \varphi(h)dh = p_0(T);$$

in particular, the right hand side is constant.

We can also obtain the formula of the period of truncation in terms of the regularized periods as in Theorem 10 of [JLR99]. We need to define the regularized integrals over $P_H\backslash H(\mathbb{A})$. For a parabolic subgroup P_H of H, let $\varphi \in \mathcal{A}_P(G)$ satisfy:

(1*) $\langle\mu, \varpi^\vee\rangle \neq \langle 2\rho^{P_H}_{Q_H} - \rho^P_Q, \varpi^\vee\rangle$, for all $Q_H \subset P_H, \mu \in \mathcal{E}_Q(\varphi)$ and $\varpi^\vee \in (\hat{\Delta}^\vee)^{P_H}_{Q_H}$;

(2*) $\langle\lambda, \alpha^\vee\rangle \neq \langle 2\rho_{P_H} - \rho_P, \alpha^\vee\rangle, \lambda \in \mathcal{E}_P(\varphi)$ for all $\alpha \in \Delta^H_{P_H}$.

We define

$$\int^*_{P_H\backslash H(\mathbb{A})} \varphi(h)\hat{\tau}_P(H_0(h) - T)dh =$$

$$\int_{K_H}\int^{\#}_{(\mathfrak{a}_P)^+_\theta}\left[\int^*_{M\backslash M(\mathbb{A})^1} \varphi(e^X mk)dm\right] e^{-\langle 2\rho_{P_H}, X\rangle}\hat{\tau}_P(X - T)dXdk. \tag{33}$$

Denote by $\mathcal{A}(G)''$ the subspace of automorphic forms $\varphi \in \mathcal{A}(G)$ that satisfy (1*) (and hence also (2*)) for all parabolic subgroups P_H of H. Clearly $\mathcal{A}(G)'' \subset \mathcal{A}(G)'$.

PROPOSITION 7.4: *If $\varphi \in \mathcal{A}(G)''$, then*

$$\int_{H\backslash H(\mathbb{A})} \Lambda_m^T \varphi(h)dh = \sum_{P_H} (-1)^{\dim((\mathfrak{a}_P)_\theta^+)} \int^*_{P\backslash H(\mathbb{A})} \varphi_P(h)\hat{\tau}_P(H_P(h)-T)dh.$$

Finally, as in [LR03], we remark that for λ_0 in the domain of convergence of $E(\varphi,\lambda)$, the regularized period $\int^*_{H\backslash H(\mathbb{A})^1} E(h,\varphi,\lambda)dh$ is well defined and bounded on the vertical strip $\operatorname{Re}\lambda = \lambda_0$.

7.3 REGULARIZED PERIODS OF CUSPIDAL EISENSTEIN SERIES. Fix a parabolic subgroup $P = MU$ of G of type $(m_1,\ldots,m_s)$. We will denote by $j = j^G$ the linear functional on $\mathcal{A}_P^1(G)$ defined by

$$j(\varphi) = \int_{K_H} \int_{M_H\backslash M_H(\mathbb{A})^1} \varphi(mk)dmdk. \tag{34}$$

Note that $j^G(\varphi) = J^G(1,\varphi,0)$, where the right hand side was defined in (18). The following is the analog of Theorem 9.1.1, the main result of [LR03].

THEOREM 7.5: *Let $\varphi \in \mathcal{A}_P^1(G)$. The regularized period*

$$\int^*_{H\backslash H(\mathbb{A})} E(h,\varphi,\lambda)dh \tag{35}$$

is zero unless $M = G$ are both of type $(n_1,\ldots,n_r,n_r,\ldots,n_1)$. Under these conditions, (35) is equal to $j(\varphi)$.

Proof: As in [LR03], the proof will follow from the distributional formula obtained in Theorem 6.3 after invoking their simple argument for tempered distributions. We first quote ([LR03], Lemma 9.1.1). The proof in our case is similar and therefore omitted.

LEMMA 7.6: *Suppose that $\phi(\lambda)$ vanishes on the hyperplanes*

$$\langle w\lambda, \varpi^\vee\rangle = \langle 2\rho_{Q_H} - \rho_Q, \varpi^\vee\rangle, \quad \text{where } w \in {}_LW_M^c \text{ and } \varpi \in \hat{\Delta}_{L_H}^H$$

for all parabolic subgroups $Q_H = L_H V_H$ of H. Then for λ_0 sufficiently regular in the positive Weyl chamber of $(\mathfrak{a}_M^G)^*$, we get

$$\int_{H\backslash H(\mathbb{A})} \theta_\phi(h)dh = \int_{\operatorname{Re}\lambda=\lambda_0} \int^*_{H\backslash H(\mathbb{A})} E(h,\phi(\lambda),\lambda)dhd\lambda. \tag{36}$$

Assume that $\phi(\lambda)$ satisfies the conditions of the lemma and further vanishes on the finitely many subspaces $((\mathfrak{a}_{M,\mathbb{C}}^G)^*)_{\xi\theta}^-$ for all $\xi \in \mathfrak{I}_M(\theta)$ such that $((\mathfrak{a}_M^G)^*)_{\xi\theta}^- \neq (\mathfrak{a}_M^G)^*$. Combining Lemma 7.6 with Theorem 6.3, we obtain as in [LR03] that

$$\int_{\operatorname{Re}\lambda=\lambda_0}\int_{H\backslash H(\mathbb{A})}^* E(h,\phi(\lambda),\lambda)dhd\lambda = \int_{\operatorname{Re}\lambda=\lambda_0} J^G(\xi,\phi(\lambda),\lambda)d\lambda$$

for λ_0 sufficiently positive, where ξ is the unique element of $W_M(\theta)$ such that $((\mathfrak{a}_M^G)^*)_{\xi\theta}^- = (\mathfrak{a}_M^G)^*$ if it exists, and the period is zero otherwise. Our analysis of minimal twisted involutions with $L_{\xi,\theta} = L$ shows that there exists $\xi \in W_M(\theta)$ such that $L_{\xi,\theta} = G$ only if $M = G$ is of the form stated in the theorem, and then of course $\xi = 1$. The argument of Lapid and Rogawski using ([LR03], Lemma 9.1.2) now takes care of the vanishing of the regularized period unless $M = G$ is of type $(n_1, \ldots, n_r, n_r, \ldots, n_1)$ and $\xi = 1$. When this is the case, the period integral is convergent and is therefore equal to the regularized period by Theorem 7.2. The period integral in this case is $j(\varphi)$. The rest of the theorem therefore follows. ∎

Since we are done with the inductive argument, for the rest of this work set $G = GL_{2n}$.

7.4 The functional equations. The functional equations satisfied by the intertwining periods were proved in ([LR03], Theorem 10.2.1). The proof is valid for our case with the usual modification, taking modulus functions into consideration. We recall the relevant results.

Theorem 7.7: *Let $\xi \in \mathfrak{I}_M(\theta)$, and let $\varphi \in \mathcal{A}_P(G)_c$. then*

- *(1) $J(\xi,\varphi,\lambda)$ extends to a meromorphic function on $((\mathfrak{a}_{M,\mathbb{C}}^G)^*)_{\xi\theta}^-$;*
- *(2) for $\xi' \in \mathfrak{I}_{M'}(\theta)$ and $w \in W(\xi,\xi')$, we have*

$$J(\xi', M(w,\lambda)\varphi, w\lambda) = J(\xi,\varphi,\lambda).$$

7.5 The period of a truncated Eisenstein series. For a θ-stable parabolic subgroup Q, we denote by v_{L_H} or also by v_{Q_H} the volume of the parallelogram

$$\Big\{\sum_{\alpha\in\Delta_{Q_H}} a_\alpha \alpha^\vee | 0 \le a_\alpha \le 1\Big\}.$$

Theorem 7.8: *Let M be a Levi subgroup of G and $\varphi \in \mathcal{A}_P^1(G)_c$. Then*

$$\int_{H\backslash H(\mathbb{A})} \Lambda_m^T E(h,\varphi,\lambda)dh = \sum_{(w,L)} v_{L_H} \frac{e^{\langle \rho_Q - 2\rho_{Q_H} + w\lambda, T\rangle}}{\prod_{\alpha\in\Delta_{L_H}}\langle \rho_Q - 2\rho_{Q_H} + w\lambda, \alpha^\vee\rangle} j(M(w,\lambda)\varphi),$$

where the sum is over all parabolic subgroups L with a type of the form $(n_1,\dots,n_r,n_r,\dots,n_1)$ with $n = n_1+\dots+n_r$, and $wMw^{-1} = L$. In particular, the period of the truncated Eisenstein series is zero unless for some permutation $w \in W(M)$, wMw^{-1} is of type $(n_1,\dots,n_r,n_r,\dots,n_1)$.

Proof: As in [LR03], we obtain

$$\int_{H\backslash H(\mathbb{A})^1} \Lambda_m^T E(h,\varphi,\lambda)dh =$$

$$\sum_{Q_H}(-1)^{\dim(\mathfrak{a}_Q)_\theta^+} \int^*_{Q_H\backslash H(\mathbb{A})^1} E^Q(h,M(w,\lambda)\varphi,w\lambda)\hat{\tau}_Q(H_Q(h)-T)dh$$

$$\begin{aligned} &= \sum_{Q_H}(-1)^{\dim(\mathfrak{a}_Q)_\theta^+} \int_{K_H}\int^{\#}_{\mathfrak{a}_{Q_H}} \left[\int^*_{L_H\backslash L(\mathbb{A})^1} E^Q(mk,M(w,\lambda)\varphi,w\lambda)dm\right] \\ &\qquad \times e^{\langle \rho_Q - 2\rho_{Q_H}+w\lambda, X\rangle}\hat{\tau}_Q(X-T)dXdk. \end{aligned} \tag{37}$$

For the integral over X, we use the formulas of [JLR99]; it equals

$$v_{L_H}\frac{e^{\langle \rho_Q-2\rho_{Q_H}+(w\lambda)_L,T\rangle}}{\prod_{\alpha\in\Delta_{L_H}}\langle \rho_Q-2\rho_{Q_H}+(w\lambda)_L,\check{\alpha}\rangle}.$$

For the inner integral we use Theorem 7.5, to get that it is zero unless $wMw^{-1} = L$ is of the required form. In the latter case every summand in (37) is of the form

$$(-1)^{\dim((\mathfrak{a}_Q)_\theta^+)}J^L(1,(e^{-\langle\rho_Q,H_Q(\cdot)\rangle}M(w,\lambda)\varphi)^{K_H}_{|L(\mathbb{A})},0).$$

The theorem then follows using (20). ∎

8. The period of the residue

In this section we prove Theorem 1.1. Proposition 7.3 plays a central roll. To apply it, we will need the following easy result.

Lemma 8.1: *Let V be a finite-dimensional vector space over $\mathbb{C}$. Let*

$$f_\lambda(T) = \sum_{i=1}^d a_i(\lambda)e^{\langle b_i(\lambda),T\rangle}$$

where $T \in V^*$, the a_i's are meromorphic functions near a point $\lambda=\lambda_0 \in V$ and the b_i's are linear endomorphisms of V such that $b_1(\lambda_0),\dots,b_d(\lambda_0) \in V$

are distinct. Fix $T \in V^*$ and assume that $\lim_{\lambda\to\lambda_0} f_\lambda(T)$ exists. Then a_i is holomorphic at λ_0 for all i and therefore

$$\lim_{\lambda\to\lambda_0} f_\lambda(T) = \sum_{i=1}^{d} a_i(\lambda_0)e^{\langle b_i(\lambda_0),T\rangle}.$$

Proof: Assume by contradiction that some a_i is not holomorphic at λ_0. Then there exists $v \in V$ such that $c \mapsto a_i(\lambda_0 + cv), c \in \mathbb{C}$ is not defined at zero. The function $c \mapsto f_{\lambda_0+cv}$ is holomorphic at zero and

$$\lim_{\lambda\to\lambda_0} f_\lambda(T) = \lim_{c\to 0} f_{\lambda_0+cv}(T).$$

We can use the Laurant expansion at zero of each of the meromorphic functions $c \mapsto a_i(\lambda_0 + cv)$ to write it as

$$a_i(\lambda_0 + cv) = \sum_{j=1}^{t} \frac{\alpha_{i,j}}{c^j} + \alpha_i(c),$$

where α_i is holomorphic at zero and there is a pair (i, j) such that $\alpha_{i,j} \neq 0$. We then get that

$$\lim_{c\to 0} e^{c\langle v,T\rangle} \sum_{i=1}^{d}\sum_{j=1}^{t} \alpha_{i,j}c^{-j}e^{\langle b_i(\lambda_0),T\rangle} \tag{38}$$

exists. Thus the limit of the Laurant polynomial defined by the double sum in (38) also exists, which in turn implies that the Laurant polynomial is zero. Thus for all j,

$$\sum_{i=1}^{d} \alpha_{i,j}e^{\langle b_i(\lambda_0),T\rangle} = 0.$$

From the linear independence of characters it now follows that $\alpha_{i,j} = 0$ for all i, j. This stands in contradiction to our assumptions. ∎

Fix a decomposition $2n = rs$ and let M be the Levi subgroup of G of type $(r, \ldots, r)$. Thus, $W(M) = W(M, M)$ is a group. Its action on the blocks of M identifies it with the permutation group $\mathfrak{S}_s$. We will view the elements of $\mathfrak{S}_s$ simultaneously as a subgroup of $W \simeq \mathfrak{S}_{2n}$ and as the group of permutations in $\{1, \ldots, s\}$. We identify $(\mathfrak{a}_M^G)^*$ with $\mathbb{R}^s$. Let $\Lambda \in (\mathfrak{a}_M^G)^*$ be defined by $\langle \Lambda, \alpha^\vee \rangle = 1$ for all $\alpha \in \Delta_M^G$. Thus,

$$\Lambda = \Big(\frac{s-1}{2}, \frac{s-3}{2}, \ldots, \frac{1-s}{2}\Big) \in \mathbb{R}^s.$$

For all $i = 1, \ldots, s-1$ we define on $\mathbb{R}^s$ the linear functional

$$R_i(\lambda) = \lambda_i - \lambda_{i+1}.$$

We will also denote

$$\mu = \rho_P - 2\rho_{P_H}.$$

Let $\varphi \in \mathcal{A}_P(G)_c$. As in [Jac84], we define the multi-residue $E_{-1}(\varphi)$ of the Eisenstein series $E(\varphi, \lambda)$ to be the limit

$$E_{-1}(g, \varphi) = \lim_{\lambda \to \Lambda} \left\{ \left[\prod_{i=1}^{s-1} (R_i(\lambda) - 1) \right] E(g, \varphi, \lambda) \right\},$$

and for $w \in \mathfrak{S}_s$ the multi-residue $M_{-1}(w)$ of the intertwining operator $M(w, \lambda)$ to be the limit

$$M_{-1}(w) = \lim_{\lambda \to \Lambda} \left\{ \left[\prod_{\{i | w(i) > w(i+1)\}} (R_i(\lambda) - 1) \right] M(w, \lambda) \right\}. \tag{39}$$

We are interested in the symplectic period of $E_{-1}(\varphi)$. We first claim that it is well defined by an absolutely convergent integral.

Lemma 8.2:

$$\int_{H \backslash H(\mathbb{A})} E_{-1}(h, \varphi) dh \tag{40}$$

is an absolutely convergent integral.

Proof: It is explained in the proof of Proposition 1 of ([JLR04], §5) how the convergence of the period of an automorphic form is only dependent on its cuspidal exponents. There, the bound of an automorphic form in terms of its cuspidal exponents, given by ([MW94], Lemma I.4.1), is used. The period of an automorphic form ϕ of G will converge if there is $\lambda \in ((\mathfrak{a}_0^P)^*)_\theta^+$, such that $\nu + \mu + \lambda$ is in the negative obtuse Weyl chamber of $(\mathfrak{a}_0^*)_\theta^+$, for the cuspidal exponents ν of ϕ. By [Jac84], $E_{-1}(\varphi)$ is concentrated at P and its only cuspidal exponent is $-\Lambda$. Note that $-\Lambda + \mu$ lies in the negative (even acute) Weyl chamber of $((\mathfrak{a}_P)^*)_\theta^+$. It is then not difficult to choose $\lambda \in ((\mathfrak{a}_0^P)^*)_\theta^+$ such that $-\Lambda + \mu + \lambda$ is in the negative obtuse Weyl chamber, i.e. it satisfies

$$\langle -\Lambda + \mu + \lambda, \varpi \rangle < 0$$

for all $\varpi \in \hat{\Delta}^H_{(P_0)_H}$. ∎

We now get from Theorem 7.2 that (40) is equal to its regularization, and from Proposition 7.3 we then get that it is the zero coefficient of the exponential polynomial in T

$$\int_{H\backslash H(\mathbb{A})} \Lambda_m^T E_{-1}(h,\varphi)dh. \tag{41}$$

In the proof of Lemma 3.1 of [Art82], pp. 47–48 it is explained why the operation of taking the multi-residue commutes with an integral of truncated Eisenstein series and with the truncation operator. After obtaining the bounds on truncated Eisenstein series, Arthur invokes Fubini's theorem to argue that the multi-residue operator commutes with the integration. His argument holds in our case for integration over $H\backslash H(\mathbb{A})$ thanks to the argument in [JLR99], pp. 190–191, where the necessary bounds are obtained for the mixed truncation of an Eisenstein series (see also Lemma 7.1 (1)). Arthur's argument for showing that the multi-residue operation commutes with the truncation operator easily modifies to argue that it commutes with mixed truncation ([Art82], pp. 47–48). We therefore obtain that (41) is equal to

$$\lim_{\lambda\to\Lambda}\left\{\left[\prod_{i=1}^{s-1}(R_i(\lambda)-1)\right]\int_{H\backslash H(\mathbb{A})} \Lambda_m^T E(h,\varphi,\lambda)dh\right\} \tag{42}$$

and that the period integral

$$\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)dh$$

is equal to the zero coefficient in the exponential polynomial (42). The first part of Theorem 1.1 follows immediately. Indeed, it follows from Theorem 7.8 that for odd s,

$$\int_{H\backslash H(\mathbb{A})} \Lambda_m^T E(h,\varphi,\lambda)dh = 0.$$

From now on we may assume s is even and denote $s = 2k$. It can easily be computed that

$$\mu = \Big(\overbrace{-\frac{1}{2},\dots,-\frac{1}{2}}^{k},\overbrace{\frac{1}{2}\dots,\frac{1}{2}}^{k}\Big).$$

Theorem 7.8 is now the identity

(43)

$$\int_{H\backslash H(\mathbb{A})^1} \Lambda_m^T E(h,\varphi,\lambda)dh = v_{P_H}\sum_{w\in\mathfrak{S}_{2k}}\frac{e^{\langle\mu+w\lambda,T\rangle}}{\prod_{\alpha\in\Delta_{P_H}}\langle\mu+w\lambda,\alpha^\vee\rangle}j(M(w,\lambda)\varphi).$$

We apply the identity (43) to (42) to obtain

(44)

$$\int_{H\backslash H(\mathbb{A})} \Lambda_m^T E_{-1}(h,\varphi)dh =$$

$$v_{P_H} \lim_{\lambda\to\Lambda} \left\{ \sum_{w\in\mathfrak{S}_{2k}} \frac{\prod_{i=1}^{2k-1}(R_i(\lambda)-1)}{\prod_{\alpha\in\Delta_{P_H}}\langle\mu+w\lambda,\alpha^\vee\rangle} j(M(w,\lambda)\varphi)e^{\langle\mu+w\lambda,T\rangle}\right\}.$$

We know that this limit exists. One may hope to compute it by computing the limit of each of the summands. Unfortunately, in general the limit of the individual summands does not exist. We will comment on that after the proof. We therefore need a bypass, using the a priori knowledge of the convergence of the limit of the sum. Some surprising cancellations play into our hands. Note that the fact that the sum converges but not the individual summands does not contradict Lemma 8.1. To see why, we remind the reader that T lies in the vector space $(\mathfrak{a}_0)_\theta^+$ and therefore the exponents of the exponential polynomial

$$\left[\prod_{i=1}^{2k-1}(R_i(\lambda)-1)\right]\int_{H\backslash H(\mathbb{A})} \Lambda_m^T E(h,\varphi,\lambda)dh$$

lie in $(\mathfrak{a}_0^*)_\theta^+$. Therefore, distinct w's may give rise to the same exponent. From (43) we see that the exponents are in the set $\{(\mu+w\lambda)_\theta^+|w\in\mathfrak{S}_{2k}\}$ and from the equality of (41) with (42) that its limit as $\lambda\to\Lambda$ exists. It therefore follows from Lemma 8.1 that

$$\int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)dh =$$

$$v_{P_H} \lim_{\lambda\to\Lambda} \sum_{\{w|(\mu+w\Lambda)_\theta^+=0\}} \frac{\prod_{i=1}^{2k-1}(R_i(\lambda)-1)}{\prod_{\alpha\in\Delta_{P_H}^H}\langle\mu+w\lambda,\alpha^\vee\rangle} j(M(w,\lambda)\varphi). \tag{45}$$

Note that as it stands, we still cannot interchange the limit with the summation in (45). Since we know that the limit exists, we may however compute it by computing a directional limit in a 'good' direction, i.e. where the limit may be computed at each summand. We need the following lemma in order to identify the Weyl elements that contribute to the sum (45). For $\sigma\in\mathfrak{S}_k$, let

$$w_\sigma(2i-1)=\sigma^{-1}(i)\le k,\quad w_\sigma(2i)=2k+1-\sigma^{-1}(i)\ge k+1.$$

LEMMA 8.3: *The correspondence $\sigma\mapsto w_\sigma$ is a bijection*

$$\mathfrak{S}_k\simeq\{w\in\mathfrak{S}_{2k}|(\mu+w\Lambda)_\theta^+=0\}.$$

Proof: It is clear that the map $\sigma \mapsto w_\sigma$ is one to one. To show it is onto, we first note that for $x = (x_1, \ldots, x_{2k}) \in \mathbb{R}^{2k} \simeq (\mathfrak{a}_M^G)^*$ we have $x_\theta^+ = 0$ if and only if $x_i = x_{2k+1-i}$ for all $i = 1, \ldots, k$. It follows that for $w \in \mathfrak{S}_{2k}$, $(\mu + w\Lambda)_\theta^+ = 0$ iff

$$w^{-1}(2k+1-i) - w^{-1}(i) = 1, \quad i = 1, \ldots, k. \tag{46}$$

Let $w \in \mathfrak{S}_{2k}$ satisfy (46). An easy inductive argument shows that $w^{-1}(i)$ must be odd for all $i \le k$, i.e. that $w(2i-1) \le k$ for all $i \le k$. Define $\sigma \in \mathfrak{S}_k$ by

$$\sigma^{-1}(i) = w(2i-1).$$

We then have $w^{-1}(2k+1-\sigma^{-1}(i)) = 1 + w^{-1}(\sigma^{-1}(i)) = 2i$, thus $w = w_\sigma$. ∎

Note that for every σ,

$$\{i \le 2k-1 | w_\sigma(i) > w_\sigma(i+1)\} = \{2, 4, 6, \ldots, 2k-2\}$$

and

$$\{i \le 2k-1 | w_\sigma(i) < w_\sigma(i+1)\} = \{1, 3, \ldots, 2k-1\}.$$

We define for all $w \in \mathfrak{S}_{2k}$ and $i = 1, \ldots, k-1$ the functionals

$$L_{w,i}(\lambda) = \lambda_{w^{-1}(i)} - \lambda_{w^{-1}(i+1)} + \lambda_{w^{-1}(2k-i)} - \lambda_{w^{-1}(2k+1-i)}$$

and

$$L_{w,k}(\lambda) = \lambda_{w^{-1}(k)} - \lambda_{w^{-1}(k+1)}.$$

If $e_i, i = 1, \ldots, 2k$ is the standard basis for $\mathbb{R}^{2k}$, for $i \le k-1, \alpha_i = e_i - e_{i+1} + e_{2k-i} - e_{2k+1-i}$ and $\alpha_k = 2(e_k - e_{k+1})$, then $\Delta_{P_H}^H = \{\alpha_i | i = 1, \ldots, k\}$ and, for each i,

$$L_{w,i}(\lambda) - \delta_{ik} = \langle \mu + w\lambda, \alpha_i^\vee \rangle.$$

We fix $v_0 \in \mathbb{R}^{2k}$, which is non-vanishing for the following finitely many hyperplanes:

$$L_{w_\sigma,i}(v_0) \neq 0, \quad 1 \le i \le k, \quad \sigma \in \mathfrak{S}_k.$$

Applying Lemma 8.3 to (45) we get that

$$\int_{H \backslash H(\mathbb{A})} E_{-1}(h, \varphi) dh =$$

$$v_{P_H} \lim_{c \to 0} \sum_{\sigma \in \mathfrak{S}_k} \left[\prod_{i=1}^{k} \frac{R_{2i-1}(v_0)}{L_{w_\sigma,i}(v_0)} \right] \left\{ \left[\prod_{i=1}^{k-1} R_{2i}(v_0) \right] c^{k-1} j(M(w_\sigma, \Lambda + cv_0)\varphi) \right\}.$$

This limit can be evaluated by taking the limit at each summand. From the definition of the multi-residue of the intertwining period we get

$$\text{(47)} \qquad \int_{H\backslash H(\mathbb{A})} E_{-1}(h,\varphi)dh = v_{P_H} \sum_{\sigma\in\mathfrak{S}_k} j(M_{-1}(w_\sigma)\varphi)\left[\prod_{i=1}^{k} \frac{R_{2i-1}(v_0)}{L_{w_\sigma,i}(v_0)}\right].$$

The right hand side is therefore independent of v_0. To complete the proof of Theorem 1.1 it is left to show that for any $\sigma_0 \in \mathfrak{S}_k$, (47) equals $v_{P_H} j(M_{-1}(w_{\sigma_0})\varphi)$. Denote $v_0 = (x_1, \dots, x_{2k})$. The expression (47) is explicitly

$$v_{P_H} \sum_{\sigma\in\mathfrak{S}_k} j(M_{-1}(w_\sigma)\varphi)$$
$$\times \frac{(x_1 - x_2)(x_3 - x_4)\cdots(x_{2k-1} - x_{2k})}{(x_{2\sigma(k)-1} - x_{2\sigma(k)}) \prod_{i=1}^{k-1}((x_{2\sigma(i)-1} - x_{2\sigma(i)}) - (x_{2\sigma(i+1)-1} - x_{2\sigma(i+1)}))}.$$

We fix $\sigma_0 \in \mathfrak{S}_k$. Since (47) is independent of v_0, we may compute it by taking the limit as $x_{2\sigma_0(k)-1} \to x_{2\sigma_0(k)}$, which is the same as cancelling out the term $(x_{2\sigma_0(k)-1} - x_{2\sigma_0(k)})$ from the top and bottom and substituting $x_{2\sigma_0(k)}$ for $x_{2\sigma_0(k)-1}$ in the expression that remains. Repeating this process consecutively for all $i = 1, \dots, k-1$, taking limits as $x_{2\sigma_0(k+1-i)-1} \to x_{2\sigma_0(k+1-i)}$, we see that for all i, (47) equals

$$v_{P_H} \sum_{\{\sigma|\sigma(j)=\sigma_0(j),k+1-i\le j\le k\}} j(M_{-1}(w_\sigma)\varphi)$$
$$\times \frac{\prod_{i\notin\{\sigma(j)|,k+1-i\le j\le k\}}(x_{2i-1} - x_{2i})}{(x_{2\sigma(k-i)-1} - x_{2\sigma(k-i)}) \prod_{i=1}^{k-1-i}((x_{2\sigma(i)-1} - x_{2\sigma(i)}) - (x_{2\sigma(i+1)-1} - x_{2\sigma(i+1)}))}.$$

Thus when $i = k-1$ the only summand that survives is the one associated with σ_0, and it is

$$v_{P_H} j(M_{-1}(w_{\sigma_0})\varphi)\frac{x_{2\sigma_0(1)-1} - x_{2\sigma_0(1)}}{x_{2\sigma_0(1)-1} - x_{2\sigma_0(1)}} = v_{P_H} j(M_{-1}(w_{\sigma_0})\varphi).$$

Theorem 1.1 is now complete. In particular, the argument above proves that $j(M_{-1}(w_\sigma)\varphi)$ is independent of σ. ∎

As promised in the introduction, we now derive the formula for the period more canonical. Let $\xi_0 \in \mathfrak{I}_M(\theta)$ be the twisted involution such that

$$\xi_0\epsilon_{2n} = \operatorname{diag}(\epsilon_{2r}, \dots, \epsilon_{2r}).$$

It is the unique minimal twisted involution $\xi \in \Xi_M(\theta)$ such that $L_{\xi,\theta} = G$. One can easily compute that for all σ,

$$w_\sigma * \xi_0 = 1_{2n}$$

and $w_1 \in W^0(\xi_0, 1_{2n})$. Recall that the functional $J(1, \varphi, \lambda) = j(\varphi)$ is independent of λ. From the functional equations of the intertwining periods, Theorem 7.7, we get that

$$J(1, M(w_1, \lambda)\varphi, 0) = J(\xi_0, \varphi, \lambda).$$

We know from the above discussion then that the limit

$$\lim_{\lambda \to \Lambda} \left[\prod_{i=1}^{k-1} R_{2i}(\lambda) \right] J(\xi_0, \varphi, \lambda) \tag{48}$$

exists and equals $j(M_{-1}(w_\sigma)\varphi)$ for each σ. We define the multi-residue $J_{-1}(\xi_0, \varphi)$ of the intertwining period $J(\xi_0, \varphi, \lambda)$ to be the limit in (48).

COROLLARY 8.4: *Using the notations of this section, in the even number of blocks case (s=2k) we have*

$$\int_{H\backslash H(\mathbb{A})} E_{-1}(h, \varphi)dh = v_{P_H} J_{-1}(\xi_0, \varphi).$$

Remark: We wish to stress here the strength of the results of Lapid and Rogawski in Proposition 7.3, and provide the simplest example where the limit in (44) cannot be computed by computing the limit inside the sum. When $n = 4$ define

$$w = (1826574) \in \mathfrak{S}_8.$$

The summand associated with this permutation is

$$\frac{(\lambda_4 - \lambda_5 - 1)}{(\lambda_4 - \lambda_8 + \lambda_5 - \lambda_1)(\lambda_3 - \lambda_7 + \lambda_6 - \lambda_2)}$$

times an expression that converges to a non-zero multiple of $j(M_{-1}(w)\varphi)$ as $\lambda \to \Lambda$. Since both linear functionals in the bottom equal zero at $\lambda = \Lambda$, the limit does not exist (not even in a 'good direction'). Using the results of Lapid and Rogawski, we were able to ignore the bad terms (which cancel each other out, since we know the limit in (44) exists) and compute the symplectic period of the residue as the zero coefficient of the exponential polynomial (44).

References

[AGR93] A. Ash, D. Ginzburg and S. Rallis, *Vanishing periods of cusp forms over modular symbols,* Mathematische Annalen **296** (1993), 709–723.

[Art81] J. Arthur, *The trace formula in invariant form,* Annals of Mathematics (2) **114** (1981), 1–74.

[Art82] J. Arthur, *On the inner product of truncated Eisenstein series,* Duke Mathematical Journal **49** (1982), 35–70.

[HR90] M. J. Heumos and S. Rallis, *Symplectic-Whittaker models for* GL_n, Pacific Journal of Mathematics **146** (1990), 247–279.

[Jac84] H. Jacquet, *On the residual spectrum of* $\mathrm{GL}(n)$, in *Lie Group Representations, II (College Park, Md., 1982/1983),* Springer, Berlin, 1984, pp. 185–208.

[JLR99] H. Jacquet, E. Lapid and J. Rogawski, *Periods of automorphic forms,* Journal of the American Mathematical Society **12** (1999), 173–240.

[JLR04] H. Jacquet, E. Lapid and S. Rallis, *A spectral identity for skew symmetric matrices,* in *Contributions to Automorphic Forms, Geometry, and Number Theory,* Johns Hopkins University Press, Baltimore, 2004, pp. 421–455.

[JR92a] H. Jacquet and S. Rallis, *Kloosterman integrals for skey symmetric matrices,* Pacific Journal of Mathematics **154** (1992), 265–283.

[JR92b] H. Jacquet and S. Rallis, *Symplectic periods,* Journal für die reine und angewandte Mathematik **423** (1992), 175–197.

[LR03] E. M. Lapid and J. D. Rogawski, *Periods of Eisenstein series: the Galois case,* Duke Mathematical Journal **120** (2003), 153–226.

[MW89] C. Moeglin and J.-L. Waldspurger, *Le spectre résiduel de* $\mathrm{GL}(n)$, Annales Scientifiques de l'École Normale Supérieure (4) **22** (1989), 605–674.

[MW94] C. Moeglin and J.-L. Waldspurger, *Décomposition spectrale et séries d'Eisenstein,* Birkhäuser Verlag, Basel, 1994. Une paraphrase de l'Écriture [A paraphrase of Scripture].

[Off] O. Offen, *Residual spectrum of* GL_{2n} *distinguished by the symplectic group,* Duke Mathematical Journal, to appear.

[Spr85] T. A. Springer, *Some results on algebraic groups with involutions,* in *Algebraic Groups and Related Topics (Kyoto/Nagoya, 1983),* North-Holland, Amsterdam, 1985, pp. 523–543.